THE BRENIER–SCHRÖDINGER PROBLEM WITH RESPECT TO FELLER SEMIMARTINGALES AND NON-LOCAL HAMILTON–JACOBI–BELLMAN EQUATIONS

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ABSTRACT. Motivated by a problem from incompressible fluid mechanics of Brenier [Bre89], and its recent entropic relaxation by [ACLZ20], we study a problem of entropic minimization on the path space when the reference measure is a generic Feller semimartingale. We show that, under some regularity condition, our problem connects naturally with a, possibly non-local, version of the Hamilton–Jacobi–Bellman equation. Additionally, we study existence of minimizers when the reference measure in a Ornstein–Uhlenbeck process.

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1. INTRODUCTION

1.1. **Main result.** We study the so-called *Brenier–Schrödinger problem* with respect to a reference measure R, that is the law of a Feller martingale on $[0,1]$. This problem consists in minimizing the relative entropy of the law P of another process Y under the marginal constraints:

(i) $Y_t \sim \mu_t$, for all $t \in [0,1]$, where (μ_t) is the data of a family of probability measure.
(ii) $(Y_0, Y_1) \sim \pi$, where π is the data of a coupling of the endpoints.

By definition of the relative entropy, every minimizer is absolutely continuous with respect to R. It particular, P is also the law of a semimartingale, and the *log-density process* Z is a semimartingale. Actually, conditionally on $\{X_0 = x\}$, Z is of the form

$$Z_t = A_{t^-} + \psi_t^x(X_t),$$

where A is an additive functional, and $(t,z) \mapsto \psi_t^x(z)$ is some function. We say that the minimizer P is *regular* provided the associated ψ^x is $\mathscr{C}^1$ in time, $\mathscr{C}^2$ in space, and the semimartingale $\psi^x_\cdot(X)$ is *special* (see definitions below). We recall that a semimartingale is special as soon as it has bounded jumps. In particular, every continuous semimartingale is special. Our main result is as follows.

Theorem. *Consider a regular solution P of the aforementioned problem, and the associated ψ^x. Then, the additive functional A is absolutely continuous. In particular, there exists a function $p\colon [0,1]\times\mathbb{R}^n \to \mathbb{R}$ such that*

$$A_t = \int_0^t p_t(X_s)\mathrm{d}s, \qquad t \in [0,1]. \tag{1.1}$$

Moreover, ψ^x is a solution to the generalized Hamilton–Jacobi–Bellman *equation*

$$\partial_t \psi^x + \mathrm{e}^{-\psi_t^x}\mathbf{A}\left(\mathrm{e}^{\psi_t^x}\right) + p_s = 0, \tag{1.2}$$

where $\mathbf{A}$ denotes the Markov generator of R.

Remark 1.1. Let us make some comments on the result:

(a) When R is a Markov diffusion, then $\mathbf{A}u(x) = \frac{1}{2}c(x)\cdot\nabla^2 u(x) + b\cdot\nabla u(x)$, for some diffusion matrix c and drift vector b. By the chain rule (1.2) becomes

$$\partial_t \psi_t^x + \mathbf{A}\psi^x + \frac{1}{2}\nabla\psi_t^x\cdot c\nabla\psi_t^x + p_t = 0,$$

which is the usual Hamilton–Jacobi–Bellman equation with pressure.

(b) The function p is interpreted as a pressure field.

(c) In the classical Hamilton–Jacobi–Bellman equation, the term $\mathbf{A}$ would have a negative sign. This phenomenon also has been observed in [ACLZ20]. It can be circumvent by considering the potential ψ associated to the time-reversed measure.

Verifying that there exist regular minimizers is an arduous task, that, in the Brownian case amounts to solve the Navier–Stokes equation (see [ACLZ20] for details). We do not take on such an accomplishment. More modestly, additionally to our main theorem, we show that there exists a minimizer, possibly non-regular, when R is the law of an Ornstein–Uhlenbeck process.

1.2. **Motivations and connections to the literature.** Understanding what happens to the Brenier–Schrödinger problem for general semimartingales, possibly with a jump part is the main motivation for this paper. The Brenier–Schrödinger problem, defined in [ACLZ20], is a relaxation of Brenier's approach [Bre89] to incompressible perfect fluids and Euler equations. This generalization, which can be seen as the entropic relaxation of Brenier original problem, aims at modelling viscous fluid dynamics. The achievements of [ACLZ20] are threefold.

(i) When the reference measure R is *Markovian*, they study the general shape of minimizers.

(ii) Whenever the reference measure R is the law of a reversible Brownian motion on $\mathbb{R}^n$ or $(\mathbb{R}\setminus\mathbb{Z})^n$, they show that minimizers of a certain form give rise to solution to the Navier–Stokes equation.

(iii) On the torus, they show that minimizers always exists, whenever $\mu_t = \mathsf{vol}$ for all t, and the coupling π has finite entropy relatively to the R_{01}.

The two last points are extended to the compact manifold setting in [GH22]. Let us stress that the particular form of the minimizer needed in [ACLZ20] is stronger than our regularity condition. Indeed, they need to assume the existence of the function p such that (1.1) holds, while (1.1) follows from our analysis. Thus, even in the purely continuous case, our result is less restrictive than that of [ACLZ20].

In the Brownian case, when the noise tends to 0, one recovers in the limit Brenier's original problem of minimization of the kinetic energy, as illustrated in [BM20]. In our possibly non-local setting, understanding the appropriate notion of small noise limit, and whether there exists a equivalent of Brenier's problem is an interesting question that could be explored in future works.

1.3. **Outline of the proof.** Our proof follows the lines of [ACLZ20]. We use, on the one hand, Itō's formula, and on the other hand, Girsanov theorem to give two different representations of the semimartingale Z. At the technical level, since Z is a special semimartingale, it thus admits a unique decomposition as a sum of a local martingale, and predictable process of finite variations. This allows us to identify the two different decompositions and conclude. In the continuous case, as for the Brownian setting of [ACLZ20], the semimartingale is always special and this assumption is unnecessary.

2. Reminders and notation

2.1. Semimartingales and their characteristics.

We refer to [JS03] for more details.

2.1.1. *Path space.* Let Ω denote the space of right-continuous with left-limit paths from $[0,1]$ to $\mathbb{R}^n$. The *canonical process* on Ω is denoted by X, that is

$$X_t(\omega) := \omega_t, \qquad \omega\in\Omega, t\in[0,1].$$

The associated *canonical filtration* defined by

$$\mathfrak{F}_t := \bigcap_{s>t}\sigma(X_r : 0\le r\le s), \qquad t\in[0,1].$$

Conveniently for $\mathsf{P} \in \mathscr{P}(\Omega)$, we write P_t for the law of X_t under P, P_{ts} for the law of (X_t, X_s) under P, and so on.

2.1.2. *Processes.* An *adapted process* is a mapping $Y : \Omega \to \Omega$ such that Y_t is $\mathfrak{F}_t$-measurable for all $t \in [0,1]$. By definition, all our processes are right-continuous with left-limit. For such Y, its *jump process* is

$$\Delta Y_t := Y_t - Y_{t-}, \qquad t \in [0,1].$$

We say that Y is *predictable* whenever seen as the mapping

$$[0,1] \times \Omega \ni (t, \omega) \mapsto Y_t(w),$$

it is measurable with respect to the σ-algebra generated by $F \times (t,s]$ for $s < t$ and $F \in \mathfrak{F}_s$. A process of *finite variation* is an adapted process $A = B - C$ with B and C two non decreasing adapted processes. In that case the *variation* of A is the process $B + C$. We also consider *generalized adapted* (resp. *predictable*) processes of the form $W : \Omega \times \mathbb{R}^n \to \Omega$.

2.1.3. *Martingales & semimartingales.* Fix $\mathsf{P} \in \mathscr{P}(\Omega)$. A *martingale* (or a P-*martingale* to emphasize the dependence on P) is an adapted process such that:

$$\mathbf{E}_{\mathsf{P}}[M_t \mid \mathfrak{F}_s] = M_s, \quad 0 \le s \le t \le 1.$$

We say that X is a *semimartingale* under P provided

$$X = X_0 + M + A, \tag{2.1}$$

where M is a P-martingale, and A is a finite variation process (P-almost surely). The decomposition (2.1) is not unique.

2.1.4. *Special semimartingales.* If A can be chosen predictable in (2.1), we say that the semimartingale is *special*. By [JS03, pp. I.4.23–24], a semimartingale is special if and only if, in (2.1), A can be chosen with P-integrable variation, and that a martingale is special as soon as its jump are bounded.

2.1.5. *Canonical jumps measure.* We consider the *canonical jumps process* and the *canonical jumps measure*

$$\Delta X_s := X_s - X_{s-}, \quad s \in [0,1];$$
$$\mu_t := \sum_{s \le t} \delta_{(s, \Delta X_s)} \mathbf{1}_{\Delta X_s \neq 0}.$$

The measure μ is a random element of $\mathscr{M}([0,1] \times \mathbb{R}^d)$, the set of Borel measures on $[0,1] \times \mathbb{R}^d$. For a, possibly random, measure $\lambda \in \mathscr{M}([0,1] \times \mathbb{R}^n)$, and a generalized process W, we write

$$(W * \lambda)_t := \int_0^t \int_{\mathbb{R}^n} W_s(y) \lambda(\mathrm{d}s \mathrm{d}y), \qquad t \in [0,1].$$

2.1.6. *Compensator of the jumps measure.* For all $\mathsf{P} \in \mathscr{P}(\Omega)$, there exists a unique predictable measure ν such that

$$[\mathbf{E}_{\mathsf{P}}[|W| * \mu] < \infty] \Rightarrow [\mathbf{E}_{\mathsf{P}}[|W| * \nu] < \infty, \text{ and } W * \mu - W * \nu \text{ is a } \mathsf{P}\text{-martingale}].$$

We call ν the compensator of μ (under P).

2.1.7. *Characteristics of a semimartingale.* We fix $h(y) := y \mathbf{1}_{|y| \le 1}$. The *big-jumps removed* version of X is

$$X^h := X - (y - h) * \mu.$$

If X is a semimartingale under P, then X^h is a special semimartingale. Thus, there exists a unique decomposition

$$X^h = X_0 + M^h + B^h,$$

with M^h a martingale, and B^h predictable. We call (B^h, C, ν) the *characteristics* of the semimartingale, where C is the quadratic variation of the continuous part of X, and ν the compensator.

Remark 2.1. In general, it is not known whether a semimartingale with prescribed characteristics exists, nor if the characteristics characterise P.

2.1.8. *Representation of semimartingales with given characteristics.* Let R be a semimartingale with characteristics (B^h, C, ν). Then, by [JS03, Thm. II.2.34], we have

$$X = X_0 + X^c + h * (\mu^X - \nu) + (y - h) * \mu^X + B^h.$$

2.2. **Markov processes and related concepts.**

2.2.1. *Shift semi-group.* We consider the *shift semi-group*

$$Y_s \circ \theta_t = Y_{t+s}, \qquad t, s \in [0, 1].$$

2.2.2. *Time reversal.* We consider the *canonical time-reversed* process

$$X_t^* := X_{1-t}, \qquad t \in [0, 1].$$

We write $\mathfrak{F}^*$ for the associated filtration.

2.2.3. *Markov process.* A process $\mathsf{P} \in \mathscr{P}(\Omega)$ is *Markov* provided,

$$\mathsf{P}(X_{t+s} \in B \mid \mathfrak{F}_t) = \mathsf{P}(X_s \in B \mid X_t).$$

We shall use the following less common alternative characterization. The process $\mathsf{P} \in$ is Markov if and only if

$$\mathsf{P}(C \cap C^* \mid X_t) = \mathsf{P}(C \mid X_t)\mathsf{P}(C^* \mid X_t), \qquad t \in [0, 1],\, C \in \mathfrak{F}_t,\, C^* \in \mathfrak{F}_t^*.$$

2.2.4. *Additive functionals.* An adapted process A is an *additive functional* provided $A_0 = 0$ and

$$A_{t+s} = A_s + A_t \circ \theta_s.$$

We identify every additive functional with the random additive set function by letting for $s < t$:

$$\begin{aligned} A_{[s,t]} &:= A_{t-s} \circ \theta_s, \\ A_{[s,t)} &:= A_{(t-s)^-} \circ \theta_s, \\ A_{(s,t]} &:= A_{(t-s)} \circ \theta_{s^-}, \\ A_{(s,t)} &:= A_{(t-s)^-} \circ \theta_{s^-}. \end{aligned}$$

2.2.5. *Reciprocal measure.* We say that P is *reciprocal* provided

$$\mathsf{P}(X_s \in B \mid \mathfrak{F}_r, \mathfrak{F}_t^*) = \mathsf{P}(X_s \in B \mid X_r, X_t), \qquad r \le s \le t.$$

As for Markov processes, we repeatedly use the alternative characterization: P is reciprocal if and only if

$$\begin{gathered} 0 \le s \le t \le 1,\, C \in \mathfrak{F}_s,\, C^* \in \mathfrak{F}_t^*,\, A \in \sigma(X_u : s \le u \le t) \Rightarrow \\ \mathsf{P}(C \cap A \cap C^* \mid X_s, X_t) = \mathsf{P}(C \cap A \mid X_s, X_t)\mathsf{P}(C^* \cap A \mid X_s, X_t). \end{gathered}$$

Clearly, all Markov processes are also reciprocal. We have that whenever P is reciprocal, then P^x is Markov.

2.3. **Feller processes and their generators.** We follow [BSW13].

2.3.1. *Feller semi-groups.* A *Feller semi-group* is a Markov semi-group $(\mathbf{T}_t)$ such that

$$\|\mathbf{T}_t u - u\|_\infty \xrightarrow[t \to 0]{} 0, \qquad u \in \mathscr{C}_0(\mathbb{R}^d).$$

2.3.2. *Infinitesimal generator.* The *infinitesimal generator* of a Feller semi-group $(\mathbf{T}_t)$ is the unique unbounded operator $\mathbf{A}$ with

$$\begin{aligned} \mathscr{D}(\mathbf{A}) &:= \left\{ u \in \mathscr{C}_0(\mathbb{R}^d) : \lim_{t \to 0} \frac{\mathbf{T}_t u - u}{t} \text{ exists} \right\}, \\ \mathbf{A}u &:= \lim_{t \to 0} \frac{\mathbf{T}_t u - u}{t}. \end{aligned}$$

The generator $\mathbf{A}$ is densely-defined and closed.

2.3.3. *Feller processes and martingales.* Let $\mathsf{P} \in \mathscr{P}(\Omega)$ be the law of a Feller process associated with the Feller semi-group $(\mathbf{T}_t)$, that is

$$\mathbf{T}_t u(x) = \mathbf{E}_{\mathsf{P}}[u(X_t) \mid X_0 = x], \qquad x \in \mathbb{R}^d,\ t \in [0,1].$$

Then, for all $f \in \mathscr{D}(\mathbf{A})$, the process

$$M_t := f(X_t) - f(X_0) - \int_0^t \mathbf{A} f(X_s)\mathrm{d}s, \qquad t \in [0,1],$$

is a P-martingale.

2.3.4. *Lévy triplet of a Feller process.* In the rest of the paper, we always assume that $\mathscr{D}(\mathbf{A})$ contains the smooth functions $\mathscr{C}_c^\infty(\mathbb{R}^d)$. In this case, by [Cou67], there exist:

(i) $\alpha : \mathbb{R}^d \to \mathbb{R}_+$;
(ii) $b^h : \mathbb{R}^d \to \mathbb{R}^d$;
(iii) $c : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ symmetric non-negative;
(iv) $N : \mathbb{R}^d \to \mathscr{M}(\mathbb{R}^d \setminus \{0\})$ with $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)N(x,\mathrm{d}y) < \infty$;

such that

$$\begin{aligned}\mathbf{A}u(x) &= -\alpha(x)u(x) + b^h(x)\cdot\nabla u(x) + \frac{1}{2}\nabla^2 u(x)\cdot c(x)\\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} (u(x+y) - u(x) - \nabla u(x)\cdot h(y))N(x,\mathrm{d}y).\end{aligned} \tag{2.2}$$

2.3.5. *Feller processes and semimartingales.* Let $\mathsf{P} \in \mathscr{P}(\Omega)$ be the law of a Feller process with Lévy triplet (b^h, c, N). Then P is a semimartingale with characteristics

$$\begin{aligned} B_t^h &= \int_0^t b^h(X_s)\mathrm{d}s;\\ C_t &= \int_0^t c(X_s)\mathrm{d}s;\\ \nu(\mathrm{d}s,\mathrm{d}y) &= N(X_s,\mathrm{d}y)\mathrm{d}s. \end{aligned} \tag{2.3}$$

2.4. **Examples of Feller semi-groups.**

2.4.1. *Heat semi-group.* Let R be the law of the Wiener process on $\mathbb{R}^d$. Consider the associated Feller semi-group:

$$\mathbf{T}_t u(x) := \int_{\mathbb{R}^d} \mathrm{e}^{-|y-x|^2/2t} \frac{\mathrm{d}y}{(2\pi t)^{d/2}}.$$

Then $\mathbf{A} = -\frac{1}{2}\Delta$. The invariant measure is the Lebesgue measure.

2.4.2. *Poisson semi-group.* Let $\mathsf{R} \in \mathscr{P}(\Omega)$ be the law of the Poisson process on $\mathbb{R}^d$ with intensity $\lambda > 0$ and jumping towards $z \in \mathbb{R}^d$. The associated Feller semi-group is

$$\mathbf{T}_t u(x) := \sum_{k\in\mathbb{N}} u(x+kz)\frac{(\lambda t)^j}{j!}\mathrm{e}^{-t\lambda}.$$

Its generator is $\mathbf{A}u(x) = \lambda(u(x+z) - u(x))$. The invariant measure is the counting measure.

2.4.3. *Symmetric stable semi-groups.* Let R be the law of the α-stable symmetric, with $\alpha \in (0,2)$. The associated Feller semi-group is

$$\mathbf{T}_t u(x) := \int u(x+y) p_{\alpha,t}(\mathrm{d}y),$$

where $p_{\alpha,t}$ satisfies

$$\widehat{p}_{\alpha,t}(\xi) = \mathrm{e}^{-t|\xi|^\alpha}.$$

Then the generator is

$$\mathbf{A}u(x) := k_\alpha \int_{\mathbb{R}^d \setminus \{0\}} (u(x+y) - u(x) - \nabla u(x) \cdot h(y)) \frac{\mathrm{d}y}{|y|^{\alpha+d}}.$$

2.4.4. *Lévy processes.* Let R be the law of a Lévy process on $\mathbb{R}^d$, that is

$$\mathbf{T}_t u := u * p_t,$$

where (p_t) is a family of infinitely divisible distributions. In this case the Lévy triplet is independent of x, and the generator is

$$\mathbf{A}u(x) := b^h \cdot \nabla u(x) + \frac{1}{2} \nabla \cdot (c \nabla u(x)) + \int_{\mathbb{R}^d \setminus \{0\}} (u(x+y) - u(x) - \nabla u(x) \cdot h(y)) N(\mathrm{d}y).$$

2.4.5. *Ornstein–Uhlenbeck stable semi-groups.* Let R be the law of the α-stable symmetric process. The associated Ornstein–Uhlenbeck process Y satisfies, under R

$$\mathrm{d}Y_t = \mathrm{d}X_t - Y_t \mathrm{d}t.$$

By solving explicitly this equation, the associated semi-group is

$$\mathbf{T}_t u(x) := \mathbf{E}_{\mathsf{R}}\left[u\left(\mathrm{e}^t x + \int_0^t \mathrm{e}^{s-t} \mathrm{d}X_s \right) \mid X_0 = x \right].$$

In this case

$$\mathbf{A}u(x) = -(-\Delta)^{\alpha/2} u(x) - x \cdot \nabla u(x).$$

The unique invariant measure is μ_α such that

$$\widehat{\mu}_\alpha(\xi) = \mathrm{e}^{-\frac{1}{\alpha}|\xi|^\alpha}.$$

2.5. **Relative entropy and Girsanov formula.**

2.5.1. *Relative entropy.* We fix $\mathsf{R} \in \mathcal{P}(\Omega)$ a semimartingale with characteristics (B^h, C, ν). The *relative entropy* of with respect to R is the functional

$$\mathcal{H}(\mathsf{P} \mid \mathsf{R}) := \begin{cases} \mathbf{E}_{\mathsf{P}}\left[\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\mathsf{R}}\right], & \text{if } \mathsf{P} \ll \mathsf{R}; \\ +\infty, & \text{otherwise.} \end{cases}$$

Elements of $\mathcal{P}(\Omega)$ that are absolutely continuous with respect to to a semimartingale are again a semimartingales [JS03]. In particular, elements with finite entropy with respect to R are semimartingales. [Léo12] characterises their characteristics.

2.5.2. *Girsanov theorem under finite entropy.* We consider the functions $\theta : \mathbb{R} \ni u \mapsto \mathrm{e}^u - u - 1$, and its convex conjugate

$$\theta^\star : \mathbb{R} \ni v \mapsto \begin{cases} (v+1)\log(v+1) - v, & \text{if } v > -1; \\ 1, & \text{if } v = -1; \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem 2.2 ([Léo12, Thms. 1 & 3]). *Then, for all* $\mathsf{P} \in \mathcal{P}(\Omega)$ *such that* $\mathcal{H}(\mathsf{P} \mid \mathsf{R}) < \infty$, *there exist an adapted process* β, *and predictable non-negative generalized process* ℓ *such that*

$$\mathbf{E}_{\mathsf{P}} \int_0^1 \beta_s \cdot C(\mathrm{d}s)\beta_s + \mathbf{E}_{\mathsf{P}} \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \theta^\star(|\ell_s(y) - 1|)\nu(\mathrm{d}s\mathrm{d}y) < \infty.$$

Moreover, P *is a semimartingale with characteristics* $(B^h + \tilde{B}, C, \ell\nu)$, *where*

$$\tilde{B}_t := \int_0^t C(\mathrm{d}s)\beta_s + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(y)(\ell_s(y) - 1)\nu(\mathrm{d}s\mathrm{d}y).$$

Remark 2.3. [Léo12] states two different theorems, one continuous martingales, and another for pure jump martingales. Decomposing the semimartingale P in its continuous part and pure jump part, it is clear that the results carry out to the mixed case.

2.5.3. *Density under finite entropy.* Additionally to the Girsanov theorem, [Léo12] obtains an expression of the density in terms of the processes β and ℓ. Consider the *log-density process*

$$Z_t := \log \mathbf{E}_{\mathsf{R}}\left[\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\mathsf{R}} \mid \mathfrak{F}_t\right].$$

Proposition 2.4 ([Léo12, Thms. 2 & 4]). *On the event* $\left\{\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\mathsf{R}} > 0\right\}$, *we have* $Z = Z^c + Z^+ + Z^-$, *where, under* P

$$\begin{aligned}
Z_t^c &:= \int_0^t \beta_s \cdot (\mathrm{d}X_s - \mathrm{d}B_s^h - C(\mathrm{d}s)\beta_s) + \frac{1}{2}\int_0^t \beta_s \cdot C(\mathrm{d}s)\beta_s;\\
Z_t^+ &:= \int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{\left\{\ell \geq \frac{1}{2}\right\}} \log \ell \, \mathrm{d}(\mu^X - \ell\nu) + \int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{\left\{\ell \geq \frac{1}{2}\right\}} \theta(\ell - 1)\,\mathrm{d}\nu\\
Z_t^- &= \int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{\left\{0 \leq \ell < \frac{1}{2}\right\}} \log \ell \, \mathrm{d}\mu^X + \int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{\left\{0 \leq \ell < \frac{1}{2}\right\}} (-\ell + 1)\,\mathrm{d}\nu.
\end{aligned}$$

Remark 2.5. (a) As noticed in [Léo12], on the event $\left\{\frac{dP}{dR} > 0\right\}$, $\ell > 0$ and the sum in the first integral of the definition of Z^- is finite.

(b) The exact value $1/2$ for the cut-off between Z^+ and Z^- is artificial and could be chosen anywhere in $(0,1)$.

(c) This expression of Z does not provide a decomposition as the sum of P-local martingale and absolutely continuous process. Indeed, the stochastic integral $(\log \ell) * (\mu^X - \ell\nu)$ is meaningless, in general. Additional assumptions could guarantee that it makes sense.

2.5.4. *Disintegration with respect to the initial condition.* So far P_0 could be arbitrary, we consider the regular disintegration of P along its marginal P_0. In this way, there exists a family $(\mathsf{P}^x)_{x\in\mathbb{R}^d}$ such that P^x is supported on $\{X_0 = x\}$ and

$$\mathsf{P} = \int \mathsf{P}^x \mathsf{P}_0(\mathrm{d}x).$$

The semimartingale property, the Markov property, the special semimartingale property, and the Feller property are stable under this conditioning.

Remark 2.6. In particular, if R is a Markov measure, then all the R^x are also Markov. The data $(\Omega, \mathfrak{F}, (X_t)_{t\in[0,1]}, (\mathsf{R}^x)_{x\in\mathbb{R}^d})$ is sometimes what is called a Markov process.

2.5.5. *Chain rule for the entropy.* By the chain rule for the entropy, [DZ10, Thm. D.13], it turns out that

$$\mathcal{H}(\mathsf{P} \mid \mathsf{R}) = \mathcal{H}(\mathsf{P}_0 \mid \mathsf{R}_0) + \int \mathcal{H}(\mathsf{P}^x \mid \mathsf{R}^x)\mathsf{P}_0(\mathrm{d}x). \tag{2.4}$$

In particular, if $\mathcal{H}(\mathsf{P} \mid \mathsf{R}) < \infty$, and that R is a semimartingale, then all the P^x $(x \in \mathbb{R}^d)$ are also semimartingales.

3. The Brenier–Schrödinger problem with respect to a Feller semimartingale

3.1. **Formulation of the problem.**

3.1.1. *Setting.* In the rest of the paper, we fix $\mathsf{R} \in \mathscr{P}(\Omega)$ a Feller semimartingale with characteristics as in (2.3). Let $\pi \in \mathscr{P}(\mathbb{R}^n \times \mathbb{R}^n)$, that we interpret as a coupling between the initial and the final position. Let $(\mu_t)_{t\in[0,1]} \in \mathscr{P}(\mathbb{R}^n)^{[0,1]}$, that we interpret as the incompressibility condition. We study the Brenier–Schrödinger minimisation problem, with respect to the measure R

$$\text{(LBS)} \qquad \inf\{\mathcal{H}(\mathsf{P} \mid \mathsf{R}) : \mathsf{P} \in \mathscr{P}(\Omega),\ \forall t \in [0,1],\ P_t = \mu_t,\ P_{01} = \pi\}$$

3.1.2. *General shape of the solutions.* By [ACLZ20, Thm. 4.7], every minimizer P in (LBS) is a reciprocal measure of the form

$$\mathsf{P} = \exp(\eta(X_0, X_1) + A_1)\mathsf{R},$$

for some Borel function $\eta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and an additive functional A. Conditioning at $\{X_0 = x\}$, we find that

$$\mathsf{P}^x = \exp(\eta(x, X_1) + A_1)\mathsf{R}^x.$$

Using the additive property of A, we find that

$$A_1 = A_{[0,1]} = A_{t-} + A_{[t,1]}.$$

It follows that the log-density process looks like

$$Z_t^x := \log \mathbf{E}_{\mathsf{R}^x}\left[\frac{\mathrm{d}\mathsf{P}^x}{\mathrm{d}\mathsf{R}^x} \mid \mathfrak{F}_t\right] = A_{t-} + \log \mathbf{E}_{\mathsf{R}^x}\left[\exp(x, X_1) + A_{[t,1]} \mid X_t\right].$$

In view of this formula, let us define

$$\psi^x(t,z) := \log \mathbf{E}_{\mathsf{R}^x}\left[\exp\big(\eta(x, X_1) + A_{[t,1]}\big) \mid X_t = z\right], \qquad t \in [0,1], z \in \mathbb{R}^d. \tag{3.1}$$

With this definition, the above expression for Z^x reads

$$Z_t^x = A_{t-} + \psi^x(t, X_t).$$

3.2. **Non-local Hamilton–Jacobi–Bellman equation.** Let us reformulate the main result of the paper with our introduced terminology and give the prove.

Theorem 3.1. *Let $\mathsf{P} \in \mathscr{P}(\Omega)$ be a regular solution to (LBS). Then, there exists a function $p : [0,1]\times\mathbb{R}^d$, such that the potential ψ^x defined in (3.1), is a strong solution to*

$$\partial_t \psi_t^x(z) + \mathrm{e}^{-\psi_t^x(z)}(\mathbf{A}\mathrm{e}^{\psi_t^x})(z) + \frac{1}{2}\nabla\psi_s^x(z) \cdot c(z)\nabla\psi_s^x(z) + p_t(z) = 0.$$

The proof follows from two different representations of Z^x as a special martingale.

3.2.1. *Decomposition using Itō's formula.* For short, write

$$J_s^x(y) := \psi_s^x(X_{s-} + y) - \psi_s^x(X_{s-}).$$

Lemma 3.2. *Under P^x, the process $\psi^x_\cdot(X)$ is a special martingale. Moreover, we have*

$$\begin{aligned}
\psi_t^x(X_t) = \psi_t^x(x) &+ \int_0^t \nabla\psi_s^x(X_{s-}) \cdot \mathrm{d}X_s^c + J * (\mu - \ell^x\nu) + \int_0^t \partial_s\psi_s^x(X_{s-})\mathrm{d}s \\
&+ \int \nabla\psi_s^x(X_{s-}) \cdot \mathrm{d}\bar{B}_s + \frac{1}{2}\int_0^t \nabla^2\psi_s^x(X^{s^-}) \cdot \mathrm{d}C_s + \int_0^t [J_s(y) - \nabla\psi_s^x(X_{s-}) \cdot h(y)]\nu(\mathrm{d}s\mathrm{d}y).
\end{aligned}$$

Proof. Since by assumption, R^x is a Feller semimartingale with characteristics (B^h, C, ν), by (2.4) and Theorem 2.2, we find that P^x is a semimartingale. Moreover, there exist an adapted process β^x

and a predictable non-negative process ℓ^x with characteristics $(B^h + \tilde{B}^x, C, \ell^x\nu)$. For short, let us consider the local martingale M, the drift $\bar{B}$, and the generalized processes J and W defined by

$$\begin{aligned} M &:= X^c + h * (\mu - \ell^x\nu);\\ \bar{B} &:= B^h + B^x;\\ J_s(y) &:= \psi_s^x(X_{s-} + y) - \psi_s^x(X_{s-});\\ W_s(y) &:= J_s(y) - \nabla\psi_s^x(X_{s-}) \cdot h(y). \end{aligned}$$

Since, by assumption, $\psi^x(X)$ is a special semi-martingale, the same argument as in [JS03, Thm. II.2.42] yields that $W * \mu - W * (\ell^x\nu) = W * (\mu - \ell^x\nu)$ is a local martingale, and

$$\begin{aligned} \psi_t^x(X_t) = \psi_t^x(x) &+ \int_0^t \partial_s\psi_s^x(X_{s-})\mathrm{d}s + \int_0^t \nabla\psi_s^x(X_{s-}) \cdot \mathrm{d}M_s + \int \nabla\psi_s^x(X_{s-}) \cdot \mathrm{d}\bar{B}_s\\ &+ \frac{1}{2}\int_0^t \nabla^2\psi_s^x(X_{s-}) \cdot \mathrm{d}C_s + W * (\mu - \ell^x\nu) + W * (\ell^x\nu). \end{aligned}$$

We conclude by observing that

$$\int_0^t \nabla\psi_s^x(X_{s-}) \cdot \mathrm{d}(h * (\mu - \ell^x\nu)) + W * (\mu - \ell^x\nu) = J * (\mu - \ell^x\nu).$$

□

3.2.2. *Decomposition using Girsanov's theorem.*

Lemma 3.3. *The process* $(\log \ell^x) * (\mu - \ell^x\nu)$ *is a well-defined local martingale. Moreover, under* P^x,

$$\begin{aligned} Z_t^x = (\log \ell^x) * (\mu - \ell^x\nu) &+ \int_0^t \beta_s^x \cdot \mathrm{d}X_s^c\\ &+ \int_0^t \int \theta^\star(\ell_s^x(y) - 1)N(X_s, \mathrm{d}y)\mathrm{d}s + \frac{1}{2}\int_0^t \beta_s^x \cdot c(X_s)\beta_s^x \mathrm{d}s, \end{aligned}$$

where the first line on the right-hand side is a local martingale, and the second line if a predictable process.

Proof. The regular solution assumption ensures that the measure P^x and R^x are equivalent. The part involving β^x follows directly from Proposition 2.4. In view of Proposition 2.4, it is sufficient to show that

$$Z_t^+ + Z_t^- = (\log \ell^x) * (\mu - \ell^x\nu) + \theta^\star(\ell^x - 1) * \nu.$$

Recall that

$$\begin{aligned} Z_t^+ + Z_t^- = &\int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{\ell^x \geq \frac{1}{2}} \log(\ell^x)\,\mathrm{d}(\mu - \ell^x\nu) + \int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{\ell^x \geq \frac{1}{2}} \theta^\star(\ell^x - 1)\,\mathrm{d}\nu\\ &+ \int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{0\leq \ell < \frac{1}{2}} \log(\ell^x)\,\mathrm{d}\mu + \int_{[0,t]\times\mathbb{R}_*^d} \mathbf{1}_{0\leq\ell^x<\frac{1}{2}}(-\ell^x + 1)\,\mathrm{d}\nu. \end{aligned} \tag{3.2}$$

Since $\ell^x\mathbf{1}_{\ell^x\geq 2}\log(\ell^x)$ and $\ell^x\,(\mathbf{1}_{1/2\leq\ell<2}\log(\ell^x))^2$ are dominated by $\theta^*(|\ell - 1|)$, which is $\mathsf{P}x$-integrable by Theorem 2.2, thus, by [JS03, II.1.27, p. 72], $\mathbf{1}_{\ell^x\geq 1/2}(\log \ell^x)$ is a local martingale. The second and the fourth terms have P^x-integrable variations because they are also dominated by $\theta^*(|\ell - 1|) * \nu$. Since Z^x is a special semimartingale, by [JS03, p. I.2.24], the third term also have P^x-integrable variations. This implies that the stochastic integral $\mathbf{1}_{0\leq\ell<1/2}\log(\ell) * (\mu^X - \ell\nu)$ is well-defined and a local martingale. Moreover by [JS03, p. II.1.28], we have

$$\mathbf{1}_{0\leq\ell^x<1/2}\log(\ell^x) * (\mu - \ell^x\nu) = \mathbf{1}_{0\leq\ell^x<1/2}\log(\ell^x) * \mu - \mathbf{1}_{0\leq\ell^x<1/2}\log(\ell^x) * (\ell^x\nu).$$

The proof is thus complete by compensating the third term in (3.2), and by using that the map $a \mapsto a * (\mu - \ell^x\nu)$ is linear. □

3.2.3. *Conclusion.*

Proof of Theorem 3.1. By the uniqueness of the decomposition of a special semimartingale, comparing the expression for Z^x in Lemmas 3.2 and 3.3, we obtain by identification of the local martingale parts

$$\log \ell^x_s(y) = \psi^x_s(X_{s-}+y) - \psi^x_s(X_{s-}), \qquad \text{and} \qquad \beta^x_s = \nabla\psi^x_\cdot(X_{s-}). \tag{3.3}$$

By identification of the predictable part, we obtain

$$\begin{aligned}\theta^\star(\ell^x-1)*\nu_s + \frac{1}{2}\int_0^t \beta^x_s\cdot C(\mathrm{d}s)\beta^x_s &= A_{s-} + \psi^x_0(x) + \int_0^t \partial_s\psi^x_s(X_{s-})\,\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \nabla^2\psi^x_s(X^{s^-})\cdot \mathrm{d}C_s + \int_0^t \nabla\psi^x_s(X_{s-})\cdot\mathrm{d}B^h_s + \int_0^t \nabla\psi^x_s(X_{s-})C(\mathrm{d}s)\beta^x_s \\ &+ \int_{[0,t]'*\mathbb{R}^d\setminus\{0\}} \nabla\psi^x_s(X_{s-})\cdot h\mathrm{d}(\ell^x-1)\nu + \int_{[0,t]\times\mathbb{R}^d_*}[J_s - \nabla\psi^x_s(X_{s-})\cdot h]\,\mathrm{d}(\ell^x\nu).\end{aligned}$$

Reporting (3.3) in the above, terms simplify and we arrive at

$$\begin{aligned}&-\int_{[0,t]\times\mathbb{R}^d_*}\left[\mathrm{e}^{J_s} - 1 - \nabla\psi^x_s(X_s)\cdot h(y)\right]\mathrm{d}\nu - \frac{1}{2}\int_0^t \nabla^2\psi^x_s(X_{s-})\cdot\mathrm{d}C_s - \frac{1}{2}\int_0^t \beta^x_s\cdot C(\mathrm{d}s)\beta^x_s \\ &= A_{t-} + \psi^x_0(x) + \int_0^t \partial_s\psi^x_s(X_{s-})\,\mathrm{d}s + \int_0^t \nabla\psi^x_s(X_{s-})\cdot\mathrm{d}B^h_s\end{aligned}$$

Recalling that the characteristics have the particular given in (2.3), we see, on the one hand, that A is actually absolutely continuous. Since A is an absolutely continuous additive functional there exist, there exists a pressure $p := [0,1]\times\mathbb{R}^d$ such that

$$A_t = \int_0^t p_s(X_s)\,\mathrm{d}s.$$

On the other hand, by differentiating with respect to t, we obtain

$$\begin{aligned}&\partial_s\psi^x_s(X_{s-}) + p_s(X_s) = -\frac{1}{2}\nabla^2\psi^x_s(X_{s-})\cdot c(X_s) - \frac{1}{2}\nabla\psi^x_s(X_{s-})\cdot c(X_s)\nabla\psi^x_s(X_{s-}) \\ &- \mathrm{e}^{-\psi^x_t(X_t)}\int_{\mathbb{R}^d_*}\left[\mathrm{e}^{\psi^x_t(X_t+y)} - \mathrm{e}^{\psi^x_t(X_t)} - \nabla\mathrm{e}^{\psi^x_t(X_t)}\cdot h(y)\right]N(X_s,\mathrm{d}y) + \nabla\psi^x_s(X_{s-})\cdot b^h(X_{s-}).\end{aligned}$$

This last equation is true P^x almost surely. To conclude from there, we use a continuity argument and the equivalence of P^x and R^x. □

4. Existence of solutions for the Ornstein-Uhlenbeck problem

Since the functional $\mathcal{H}(\cdot\mid\mathsf{R})$ is convex and lower semi-continuous on $\mathcal{P}(\Omega)$, by the direct method of the calculus of variations, we obtain immediately the following result.

Lemma 4.1. *Assume that there exist* $\mathsf{P}\in\mathcal{P}(\Omega)$, *such that* $\mathsf{P}_{01}=\pi$, $\mathsf{P}_t=\mu_t$ *for all* $t\in[0,1]$, *and* $\mathcal{H}(\mathsf{P}\mid\mathsf{R})<\infty$. *Then, there exists a unique minimizer to* (LBS).

Thus, in relation with Theorem 3.1, we have to answer two questions:

(A) Can we find a candidate $\mathsf{P}\in\mathcal{P}(\Omega)$ for (LBS) with finite entropy?
(B) Is the unique solution to (LBS) regular?

[ACLZ20, Prop. 6.1] gives a positive answer to (A), when R is the reversible Brownian motion on the torus, $\mu_t=\mathsf{vol}$ for all $t\in[0,1]$ (incompressible case), and π is any coupling satisfying $\mathcal{H}(\pi\mid\mathsf{R}_{01})<\infty$. [GH22] studies the case of the reflected Brownian measure on some quotient spaces, and of a non-reversible Brownian measure in $\mathbb{R}^n$. In the latter case, the incompressible condition translates to Gaussian marginal constraints. All the results motioned above, the necessary and sufficient condition for existence is $\mathcal{H}(\pi|\mathsf{vol}^2)<+\infty$. In the Ornstein-Uhlenbeck case, we only manage to prove existence when π is a Gaussian of a certain form.

In this section, we obtain a positive answer to (A) when R is a reversible Brownian Ornstein–Uhlenbeck process on $\mathbb{R}$. In the language of semimartingales , R as characteristics $(-X_t, \sqrt{2}, 0)$, and is started from a Gaussian distribution. In terms of stochastic differential equations, under R, the canonical process X satisfies

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}\mathrm{d}W_t - X_t\mathrm{d}t, \\ X_0 \sim \gamma := \mathcal{N}(0,1), \end{cases}$$

where W is an Brownian motion, under R, independent of X_0. The measure γ is the unique invariant measure of this process, thus we study (LBS) under the natural incompressible condition $\mu_t := \gamma$ for all $t \in [0,1]$. Hence, we consider the minimisation problem Namely, our minimisation problem in this specific case only depends on the parameter $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$, and reads

$$\inf\{\mathcal{H}(\mathsf{P} \mid \mathsf{R}) : \mathsf{P} \in \mathcal{P}(\Omega), \mathsf{P}_t = \gamma \,\forall t \in [0,1], \mathsf{P}_{01} = \pi\}. \tag{4.1}$$

For $|c| \leq 1$, let us write

$$\gamma_c := \mathcal{N}\left(0, \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}\right)$$

Our results in this case is as follows.

Proposition 4.2. *Let $\pi := \gamma_c$ with $4e^{-1} - 3e^{-1/3} \leq c < 1$. Then, the problem (4.1) admits a unique solution.*

Remark 4.3. Since $\mathsf{P}_0 = \mathsf{P}_1 = \gamma$, we necessarily have that the variance under π is 1. Also if $|c| := 1$, then π is degenerated and $\mathcal{H}(\pi|\mathsf{R}_{01})$ is not finite and so the problem would not have any solution. Lastly, for the particular $c := e^{-1}$, since $\pi = \mathsf{R}_{01}$, the problem admits the trivial solution $P = R$.

As in [ACLZ20; GH22], we create candidate path measures as mixture of R-bridges. In the setting of the Ornstein-Uhlenbeck process, we exploit the following explicit representation for the bridge

$$\mathsf{R}^{xy} := \mathsf{R}(\cdot \mid X_0 = x, X_T = y), \qquad x, y \in \mathbb{R},\ T \in [0,1].$$

Lemma 4.4 ([BK13]). *The Ornstein–Uhlenbeck bridge R^{xy} coincides with the law of the process*

$$U_t^{x,y} := \frac{\sinh(T-t)}{\sinh(T)}x + \frac{\sinh(t)}{\sinh(T)}y + \sqrt{2}\int_0^t \frac{\sinh(T-t)}{\sinh(T-s)}\,\mathrm{d}W_s,$$

where W is a standard Brownian motion. In particular,

$$\mathsf{R}_t^{xy} = \mathcal{N}\left(\frac{\sinh(T-t)}{\sinh(T)}x + \frac{\sinh(t)}{\sinh(T)}y, 2\frac{\sinh(T-t)\sinh(t)}{\sinh(T)}\right).$$

Let $T > 0$ and $\sigma \in \mathcal{P}(\mathbb{R}^2)$. We define

$$\mathsf{Q} := \int_{\mathbb{R}^2} \mathsf{R}^{xy}\sigma(\mathrm{d}x\mathrm{d}y). \tag{4.2}$$

The path measure Q is a mixture of Ornstein–Uhlenbeck bridges.

Remark 4.5. For Brownian bridges, [ACLZ20; GH22] can choose for σ a product measure such that the mixture Q satisfies the incompressibility condition, that is $\mathsf{Q}_t = \mathrm{vol}$ or $\mathsf{Q}_t = \mathcal{N}(0, 1/4)$, for all $t \in [0,1]$. For Ornstein-Uhlenbeck bridges, choosing σ as a product *cannot* yield an invariant process Q. This explains why we need to introduce correlations, and why we are this limited to Gaussian couplings for π.

Lemma 4.6. *Consider the bridge mixture Q as defined in (4.2), with $\sigma := \gamma_\rho$ for some $|\rho| < 1$. Then, $\mathsf{Q}_t = \gamma$ for all $0 \leq t \leq T$ if and only if $\rho = \mathrm{e}^{-T}$.*

Proof. If $\rho = e^{-T}$, then $\sigma = \mathsf{R}_{0T}$ and $\mathsf{Q} = \mathsf{R}$. Let us show that it is the only possible ρ. Let $0 \leq t \leq T$. According to Lemma 4.4 and the definition of Q, Q_t is the law of

$$\frac{\sinh(T-t)}{\sinh(T)}X + \frac{\sinh(t)}{\sinh(T)}Z + \sqrt{\frac{\sinh(T-t)\sinh(t)}{\sinh(T)}}W,$$

where $(X, Z) \sim \gamma_\rho$ and W is an independent standard Gaussian random variable. In particular, we find for the variance

$$\mathbf{Var}[Q_t] = \frac{\sinh^2(T-t)}{\sinh^2(T)} + 2\rho\frac{\sinh(T-t)\sinh(t)}{\sinh^2(T)} + \frac{\sinh^2(t)}{\sinh^2(T)} + 2\frac{\sinh(T-t)\sinh(t)}{\sinh(T)}.$$

So, the variance is constant and equals 1 if and only if for all $0 < t < T$, we have

$$2\rho\sinh(T-t)\sinh(t) = \sinh^2(T) - \sinh^2(T_t) - \sinh^2(t) - 2\sinh(T)\sinh(T-t)\sinh(t).$$

By direct computations, the right-hand side becomes

$$2\sinh(T-t)\sinh(t)(\cosh(T) - \sinh(T)).$$

Thus, the variance is constant and equals to 1 if and only if $\rho = \cosh(T) - \sinh(T) = \mathrm{e}^{-T}$. □

Proof of Proposition 4.2. Actually, we need to concatenate several bridges in order to conclude. In this way, we obtain a free parameter for us to optimise. We let $r := e^{-1/3}$, and $s \in \mathbb{R}$ to be chosen later. Let $\sigma \in \mathscr{P}(\mathbb{R}^4)$ be the centred Gaussian law with covariance

$$C := \begin{pmatrix} 1 & r & s & c \\ r & 1 & r & s \\ s & r & 1 & r \\ c & s & r & 1 \end{pmatrix},$$

and $\mathsf{Q} \in \mathscr{P}(\Omega)$ defined by

$$\mathsf{Q} := \int_{\mathbb{R}^3} \mathsf{R}(\cdot \mid X_0 = x, X_{1/3} = u, X_{2/3} = v, X_1 = y)\sigma(\mathrm{d}x\mathrm{d}u\mathrm{d}v\mathrm{d}y).$$

The measure Q has finite relative entropy. By the chain rule for the entropy (2.4), we have

$$\mathcal{H}(\mathsf{Q} \mid \mathsf{R}) = \mathcal{H}(\pi \mid \mathsf{R}_{01}) + \int \mathcal{H}(\sigma^{xy} | \mathsf{R}^{xy}_{1/3,2/3})\pi(\mathrm{d}x\mathrm{d}y).$$

Since, π and R_{01} on the one hand, and σ^{xy} and $\mathsf{R}^{xy}_{1/3,2/3}$ on the other hand, are non-degenerated Gaussian laws, their relative entropies are finite. Furthermore, $\mathcal{H}(\sigma^{xy} \mid \mathsf{R}^{xy}_{1/3,2/3})$ is a quadratic polynomial in x and y. Thus it is integrable with respect to the Gaussian measure π.

The measure Q satisfies the marginal conditions. By construction we have that $\mathsf{Q}_{01} = \pi$. Let $0 < t < 1$. Since R is a reciprocal measure, whenever $h \in \{0, 1/3, 2/3\}$

$$\mathsf{Q}_t = \int \mathsf{R}_t(\cdot \mid X_h = x, X_{h+1/3} = y)\gamma_r(\mathrm{d}x\mathrm{d}y),$$

Hence, using Lemma 4.6, we have $\mathsf{Q}_t = \gamma$.

Handling the parameters. To conclude, let us derive conditions on s and c, under which C is a covariance matrix, that is positive definite. Since C is a Toeplitz matrix, its eigenvalues are

$$\frac{1}{2}(c + r + 2 \pm \sqrt{c^2 - 2cr + 5r^2 + 8rs + 4s^2}),$$
$$\frac{1}{2}(-c - r + 2 \pm \sqrt{c^2 - 2cr + 5r^2 - 8rs + 4s^2}).$$

Thus, C is a covariance matrix if and only if

$$\frac{s^2 + 2rs + r^2 - r - 1}{r + 1} < c < \frac{s^2 - 2rs + r^2 + r - 1}{r - 1}.$$

These two inequalities have solutions if and only if $s \in (2r^2 - 1, 1)$. Then for each $r(4r^2 - 3) < c < 1$, there exists $s \in (2r^2 - 1, 1)$ such that Γ is the covariance matrix of a non-degenerated Gaussian measure. This proves the existence of a unique solution. □

Remark 4.7. Our candidate measure Q is slightly more involved than the one from [ACLZ20], where the bridge is only conditioned at the time $1/2$. In our case, their approach would only prove the existence of solutions for $2e^{-1} - 1 \leq c < 1$. Conditioning at times $1/3$ and $2/3$ gives more flexibility, thanks to the free parameter s.

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