The Annals of Probability
2024, Vol. 52, No. 3, 1162–1200
https://doi.org/10.1214/24-AOP1689

SUPERCONVERGENCE PHENOMENON IN WIENER CHAOSES

BY RONAN HERRY[1,a], DOMINIQUE MALICET[2,c] AND GUILLAUME POLY[1,b]

[1]*IRMAR, Université de Rennes,* [a]*ronan.herry@univ-rennes.fr,* [b]*guillaume.poly@univ-rennes.fr*
[2]*LAMA, Université Guistave Eiffel,* [c]*dominique.malicet@univ-eiffel.fr*

We establish, in full generality, an unexpected phenomenon of strong regularization along normal convergence on Wiener chaoses. Namely, for every sequence of chaotic random variables, convergence in law to the Gaussian distribution is automatically upgraded to *superconvergence*: the regularity of the densities increases along the convergence, and all the derivatives converge uniformly on the real line. Our findings strikingly strengthen known results regarding modes of convergence for normal approximation on Wiener chaoses. Without additional assumptions, quantitative convergence in total variation is established by Nourdin and Peccati (*Probab. Theory Related Fields* **145** (2009) 75–118), and later on amplified to convergence in relative entropy by Nourdin, Peccati and Swan (*J. Funct. Anal.* **266** (2014) 3170–3207).

Our result is then extended to the multivariate setting and for polynomial mappings of a Gaussian field, provided the projection on the Wiener chaos of maximal degree admits a nondegenerate Gaussian limit. While our findings potentially apply to any context involving polynomial functionals of a Gaussian field, we emphasize, in this work, applications regarding: improved Carbery–Wright estimates near Gaussianity, normal convergence in entropy and in Fisher information, *superconvergence* for the spectral moments of Gaussian orthogonal ensembles, moments bounds for the inverse of strongly correlated Wishart-type matrices, and *superconvergence* in the Breuer–Major Theorem.

Our proofs leverage Malliavin's historical idea to establish smoothness of the density via the existence of negative moments of the Malliavin gradient, and we further develop a new paradigm to study this problem. Namely, we relate the existence of negative moments to some explicit spectral quantities associated with the Malliavin Hessian. This link relies on an adequate choice of the Malliavin gradient, which provides a novel decoupling procedure of independent interest. Previous attempts to establish convergence beyond entropy have imposed restrictive assumptions ensuring finiteness of negative moments for the Malliavin derivatives Our analysis renders these assumptions superfluous.

The terminology *superconvergence* was introduced by Bercovici and Voiculescu (*Probab. Theory Related Fields* **103** (1995) 215–222) for the central limit theorem in free probability.

CONTENTS

Received May 2023; revised January 2024.
MSC2020 subject classifications. 60B12, 60H07, 28C20.
Key words and phrases. Wiener chaos, Malliavin calculus, Malliavin–Stein approach, superconvergence.

1. Introduction.

1.1. *Summary of the results.* Controlling the regularity of a sequence of asymptotically normal random variables is a prevalent question in probability theory. In the framework of the usual central limit theorem, the smoothing effect of convolution entails the following regularization phenomenon. Let (X_i) be a sequence of centred, normalized, and i.i.d. random variables such that $\mathbf{E}[\mathrm{e}^{\mathrm{i}tX_1}] \sim t^{-\theta}$ as $t \to \infty$ for some $\theta > 0$; then for all $q \in \mathbb{N}$, there exists n large enough such that the law of $n^{-1/2}\sum_{i=1}^{n} X_i$ has a density with respect to the Lebesgue measure that is $\mathscr{C}^q$ and converges in the $\mathscr{C}^q$-topology to the Gaussian density, $\mathscr{C}^q$ being the space of functions $f : \mathbb{R} \to \mathbb{R}$ with $f, f', \dots, f^{(q)}$ continuous and bounded, equipped with the topology induced by the norm $\|f\|_{\mathscr{C}^q} := \|f\|_\infty + \cdots + \|f^{(q)}\|_\infty$.

Extending normal convergence to nonlinear functionals of a random field, in particular polynomial functionals of a Gaussian field, is a fertile and lively area of research. We refer to

[36] and the references therein as well as to [32] for an overview. Despite numerous results regarding normal approximation, capturing the above regularization phenomenon for Gaussian polynomials has so far remained out of reach: the best known modes of convergence are the total variation distance [35] or the relative entropy [39]. As opposed to the central limit theorem, thoroughly covered by [28], due to the absence of convolution, questions regarding regularity in this nonlinear framework are much more challenging.

In this article we develop a novel approach to study the regularity of nonlinear functionals of a Gaussian field, based on Malliavin calculus and Wiener chaoses theory. In this setting we show regularization of densities along normal convergence. This discovery drastically strengthens the aforementioned results. Before stating our results, we recall that the Wiener chaoses are the infinite-dimensional counterpart of the well-known Hermite polynomials. In particular, they form an orthogonal basis with respect to the Wiener measure. We also recall that nonconstant random variables in a finite sum of Wiener chaoses always admit a density with respect to the Lebesgue measure [45]. We give more details on Wiener chaoses in Section 3. We write d_{FM} for the *Fortet–Mourier* distance; it metrizes the topology of convergence in law. In the statement below, the Fortet–Mourier distance plays no specific role and could be replaced by any distance metrizing the topology of convergence in law. We also write $\mathbb{N}$ for the set of natural integers, and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Our main result reads as follows.

THEOREM 1. *Let $d \in \mathbb{N}^*$ and $q \in \mathbb{N}$, there exist $\delta = \delta_{q,d} > 0$ and $C = C_{q,d} > 0$ such that for all F in the Wiener chaos of degree d, with density f, we have*

$$d_{\mathrm{FM}}\big(F, \mathcal{N}(0,1)\big) \leq \delta \Rightarrow \big[f \in \mathscr{C}^q \text{ and } \|f\|_{\mathscr{C}^q} \leq C\big].$$

Closely related to Theorem 1, is the following sequential theorem that gives the announced regularization phenomenon along normal convergence on Wiener chaoses. Write φ for the standard Gaussian density.

THEOREM 2. *Let (F_n) be a sequence of random variables in a Wiener chaos of fixed degree, with respective density (f_n). Then*

$$F_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0,1) \Leftrightarrow \big[\|f_n^{(q)} - \varphi^{(q)}\|_\infty \xrightarrow[n\to\infty]{} 0, \forall q \in \mathbb{N}\big].$$

In the above theorem, the quantity $f_n^{(q)}$ is only defined for n large enough.

Our approach to regularity of laws on the Wiener space originates from Malliavin's seminal contribution [30]. In this paper, Malliavin shows that a random variable F on the Wiener space has a smooth law, provided the norm of its *Malliavin derivative* $\Gamma[F,F] := \|\mathsf{D}F\|^2$ admits negative moments at every order. Our Theorems 1 and 2 proceed from Malliavin's strategy together with the following result, that is the pivotal tool of this paper.

THEOREM 3. *Let $(F_n)_{n\geq 1}$ be a sequence of random variables in a Wiener chaos of fixed degree. Assume that (F_n) converges in law to $\mathcal{N}(0,1)$, then*

$$\limsup_{n\to\infty} \mathbf{E}\big[\Gamma[F_n,F_n]^{-q}\big] < \infty, \quad q \in \mathbb{N}.$$

By standard Malliavin calculus techniques, which are recalled in Section 3.2.2, Theorem 1 and Theorem 2 follow from Theorem 3. Establishing Theorem 3 is the main contribution of this paper. The proof of Theorem 3 is conducted in Section 6 and relies on the following key ideas:

- Proposition 31 shows that Theorem 3, stated for scalar random variables, is actually equivalent to its version for vector-valued random variables.

- Thanks to a new representation of Malliavin derivatives, we relate, in Section 5.2, the negative moments of $\Gamma(F, F)$ to spectral quantities associated to the Malliavin Hessian.
- For chaos of degree 2, the Malliavin Hessian is deterministic and the control of these spectral quantities is straightforward (see Section 4.2).
- For chaos of degree ≥ 2, we proceed by induction. We first compare the Malliavin Hessian with the *compressed Malliavin Hessian* which is obtained by multiplying it by a large independent Gaussian matrix of small rank, which enables us to reduce the dimension (see Section 6.4). Then Section 6.5 we interpret the compressed Hessian as a vector of random variables in a chaos of degree $d-1$ allowing to conclude by the induction hypothesis.

We give a more detailed summary of our approach in Section 1.4.

1.2. *Compendium of related results.*

1.2.1. *Central limit on the Wiener space and the Fourth Moment Theorem.* The breakthrough by Nualart and Peccati [43] provides an efficient and tractable criterion to establish normal convergence on Wiener chaoses. Their *Fourth Moment Theorem* states that a sequence in a Wiener chaos of fixed degree converges in law to a Gaussian if and only if the sequences of its second and fourth moments converge to the respective moments of the target Gaussian distribution. This result has stemmed a new line of research establishing simple, yet powerful, conditions for normal convergence on the Wiener space. Among the most notable developments regarding limit theorems on Wiener chaoses, let us mention the following nonexhaustive contributions:

[44] Peccati and Tudor extend the fourth moment theorem to random vectors whose each coordinate lives in a Wiener chaos, possibly of different degrees.

[42] Ortiz-Latorre and Nualart establish that a sequence of random variables (F_n) in a fixed Wiener chaos converge in law to a Gaussian if and only if $\Gamma[F_n, F_n]$ converge to a constant in L^2.

[35] Nourdin and Peccati combine Stein's method and Malliavin calculus in order to obtain a quantitative fourth moment theorem. Namely, for a chaotic random variable F with $\mathbf{E}[F^2] = 1$, we have

$$d_{\mathrm{TV}}\big(F, \mathcal{N}(0,1)\big) \leq c\,\mathbf{Var}\big[\Gamma[F,F]\big]^{1/2} \leq c\,\mathbf{E}\big[F^4 - 3\big]^{1/2}.$$

This landmark contribution emphasizes the symbiotic interplay between Stein's method and Malliavin calculus: on the Wiener space, Stein kernels, that quantify convergence in distribution, are explicitly computable through integration by parts for the Malliavin operators; see [32] for a regularly updated list of contributions in this area.

[3, 26] Ledoux, and Azmoodeh, Campese and Poly leverage the rich spectral properties of Wiener chaoses to revisit the fourth moment theorem. This approach avoids the intricate product formula for Wiener chaoses and insists instead on moment inequalities for chaotic random variables. For further developments of this strategy, see [4, 29]

[33] Nourdin, Peccati and Swan improve further the Malliavin–Stein approach by establishing a fourth moment bound for the relative entropy with respect to the Gaussian measure. In view of Pinsker's inequality, this improves the convergence in total variation of [35], although the rate of convergence in [33] are nonsharp by an additional logarithmic factor.

All the results presented above hold for a *general sequence of chaotic random variables*, that is, they hold without any further assumption on the sequence. For such general sequences, until the present contribution, no results beyond convergence in entropy were available.

1.2.2. *Controlling the regularity via the negative moments of the Malliavin gradient.* Originally, Malliavin [30] uses controls on negative moments of the Malliavin derivative to give a new, purely probabilistic, proof of Hörmander theorem on hypoelliptic operators [19]; see also the recent selfcontained survey [16]. Since then, establishing that $\Gamma[F, F]^{-p} \in L^1$ has become a practical criterion in the study of the regularity of the density of F. For instance, in various contexts the recent works [2, 12, 23, 33] implement this strategy. In this paper as well as in the companion paper [17], we propose a new general estimate on the negative moments of $\Gamma[F, F]$, involving the spectrum of the Hessian matrix of F. We then bring the aforementioned fine results regarding normal convergence on Wiener chaoses, arising from the Malliavin–Stein method, to bear on establishing existence of negative moments for asymptotically normal chaotic sequences.

Previous works on the Wiener space have implemented the strategy of controlling negative moments of the Malliavin derivative to improve normal convergence. These various attempts fail to capture the generality of the phenomenon we exhibit in this work and are constrained by unnecessary assumptions in order to carry their analysis. Let us mention the most prominent developments in that regard. In the three following examples, the present contribution renders the additional assumptions on the negative moments of the Malliavin derivative unnecessary.

[22] Assuming negative moments for the Malliavin derivative, Hu, Lu and Nualart give a $\mathscr{C}^\infty$ version of the celebrated bound of [35]. Namely, take a sequence (F_n) of chaotic random variables with variance 1 and such that

$$\limsup_{n\to\infty} \mathbf{E}\big[\Gamma[F_n, F_n]^{-p}\big] < \infty, \quad p \in \mathbb{N},$$

they show the following Malliavin–Stein bound for *superconvergence*

$$\| f_n^{(q)} - \varphi^{(q)} \|_\infty \leq c_q\, \mathbf{E}\big[F_n^4 - 3\big]^{1/2}, \quad q \in \mathbb{N}.$$

[33] In the same spirit, under the assumptions of negative moments, Nourdin and Nualart establish a fourth moment theorem in relative Fisher information. The authors are, moreover, able to apply their criterion to general sequences of random variables living in the *second* Wiener chaos. In this case an explicit diagonalization argument allows to conclude on the existence of negative moments. We also refer to [27], Proposition 5.5, for related bounds on the negative moments of the Malliavin derivatives in connection with the Fisher information.

[23] Hu, Nualart, Tindel and Xu establish *superconvergence* to the normal distribution of properly rescaled Hermite sums of a stationary Gaussian field, under the assumption that the spectral measure admits a density whose logarithm is integrable, together with mild additional assumptions on the spectral measure.

1.2.3. *Regularity for general polynomials in Gaussian variables.* Our techniques strongly profit from the asymptotic normality of the sequence under consideration. The question of the regularity of the law for a generic element of a Wiener chaos, possibly away from normality, has attracted several important contributions. For instance, [8] establishes that the law of a nonconstant polynomials in independent Gaussian variables always belong to a fractional Nilkolskii–Besov space. Moreover, they show that this regularity is the best possible at this level of generality; see also the survey [7] and the references therein.

1.2.4. Superconvergence *in free probability.* In [5] Bercovici and Voiculescu discover a remarkable regularization in the free central limit theorem: indeed, for *any* free and identically distributed random variables (X_n) the law of $n^{-1/2} \sum_{i=1}^n X_i$ is eventually smooth, and the sequence of respective densities converges to the semicircular density, in the sense of uniform convergence on compact sets of all the derivatives. They call this better-than-expected convergence "*superconvergence*," and we borrow the terminology from their work.

1.3. *Detailed review of the results.* As anticipated, we establish a regularization phenomenon along normal convergence on Wiener chaoses. Our techniques exploit the rich structure of Wiener chaoses and yield existence of negative moments for $\|\mathsf{D}F_n\|$, as soon as (F_n) converges in law to a nondegenerate Gaussian. This phenomenon has gone unnoticed until now. It allows, in particular, an important enhancement of the normal convergence on Wiener chaoses: from total variation [35] or relative entropy [39] to *superconvergence*, that is, uniform convergence of the densities as well as all their derivatives. From [35, 42] normal convergence of (F_n) guarantees that $(\|\mathsf{D}F_n\|)$ converges to a constant in L^p $(p \geq 1)$. Here we refine this information on the behaviour of the Malliavin derivatives and provide analogous results for negative p. This enables Malliavin calculus techniques to establish regularization. We actually obtain a version of our result for sequences of vectors whose coordinates are in Wiener chaoses, possibly of different degrees, and some variations of the result which hold for finite sums of Wiener chaoses. We discuss below various applications.

1.3.1. *Regularization on Wiener chaoses.* In what follows, we denote by $\mathcal{W}_m$ the mth Wiener chaos associated to a fixed Gaussian field and Γ the associated *square field* operator, that is, $\Gamma[F, F] := \|\mathsf{D}F\|^2$ where D is the *Malliavin derivative*. Whenever, $\vec{F} = (F_1, \ldots, F_d)$ is vector-valued, we consider the *Malliavin matrix*

$$\Gamma(\vec{F}) := \big(\Gamma[F_i, F_j]\big)_{ij}.$$

Following Malliavin's idea, our regularization results are obtained through the existence of negative moments for the Malliavin matrix. The general version of our theorem for random vectors is as follows.

THEOREM 4. *Let d and $m_1, \ldots, m_d \in \mathbb{N}^*$. Consider a sequence $(\vec{F}_n) \subset \prod_i \mathcal{W}_{m_i}$ such that*

$$\vec{F}_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, I_d).$$

Then for every $q \geq 1$, there exist $N \in \mathbb{N}$ and $c > 0$ such that

$$\mathbf{E}\big[\det \Gamma(\vec{F}_n)^{-q}\big] \leq c, \quad n \geq N. \tag{1}$$

REMARK 5. Equivalently, the sequence $(\Gamma(\vec{F}_n)^{-1})_{n\geq 1}$ is asymptotically bounded in L^q, in the sense that, for any matrix norm $\|\cdot\|$,

$$\limsup_{n\to\infty} \mathbf{E}\big[\|\Gamma(\vec{F}_n)^{-1}\|^q\big] < \infty.$$

REMARK 6. We can consider more general limiting laws $\mathcal{N}(0, C)$ with C an invertible $d \times d$ matrix. Without loss of generality, assume that (m_i) is not decreasing. Let $k \leq n$ and $i_1, \ldots, i_k$ be the indices such that $m_{i_l} \neq m_{i_l+l}$. Then since elements of a Wiener chaos of different degrees are uncorrelated [36], Proposition 2.7.5, we find that C_n, the covariance matrix of $\vec{F}_n$, is diagonal by blocks and contains exactly k blocks. Write $C_n[1], \ldots, C_n[k]$ for those blocks. The block $C_n[l]$ is a square matrix of size $n_l \times n_l$, where n_l is the numbers of indices i such that $m_i = m_{i_l}$. Since C is invertible, for n large enough C_n is also invertible. Inverting the matrix by block and using that Wiener chaos are stable by nonzero linear combinations, we obtain that $C_n^{-1/2}\vec{F}_n$ satisfies the assumptions on the theorem.

REMARK 7. It is possible to consider a slight generalization of our result, where we consider degrees $m_i^{(n)}$ that depend on n but are uniformly bounded. In this case the sequence

$(m_i^{(n)})$ assume only finitely many distinct values. Thus, we can extract finitely many subsequences that satisfy the assumptions of the theorem. It would be interesting, and more difficult, to consider varying degrees $m_i^{(n)}$ with $m_i^{(n)} \to \infty$, possibly with some prescribed speed of divergence. We do not how to tackle this problem; it would require the demanding and involved task of tracking quantitatively the dependence in the degrees of the chaos in all our estimates.

As anticipated, an important consequence of Theorem 4 is a *superconvergence* phenomenon on Wiener chaoses: by integration by parts, negative moments for $\det \Gamma(\vec{F})$ yield regularity estimates on the density. We measure the regularity in the Sobolev space $W^{q,p}(\mathbb{R}^d)$ with $p \in [1, \infty]$, and $q \in \mathbb{N}$. This is the space of (class of equivalence of) functions $f : \mathbb{R}^d \to \mathbb{R}$ that are in L^p such that

$$\partial^\alpha f := \frac{\partial^q f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} \in L^p, \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d,\ |\alpha| := \alpha_1 + \dots + \alpha_d = q.$$

Here the partial derivatives are taken in the sense of distributions. We often omit the dependence on $\mathbb{R}^d$ in our notation. We equip $W^{p,q}$ with the Sobolev norm

$$\|f\|_{W^{q,p}} := \sum_{|\alpha| \le q} \|\partial^\alpha f\|_{L^p}.$$

For $q = 0$, our definition is understood as $W^{0,p} = L^p$. Accordingly, we extend our definition of $\mathscr{C}^q$ as the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ with all the partial derivatives $\partial^\alpha f$ for $|\alpha| \le q$ continuous and bounded. It is equipped with the norm

$$\|f\|_{\mathscr{C}^q} := \sum_{|\alpha| \le q} \|\partial^\alpha\|_\infty.$$

We recall that, by the Sobolev embeddings [20], Section 4.5,

$$\|\cdot\|_{W^{q,1}} \le \|\cdot\|_{W^{q,p}} \le c\|\cdot\|_{W^{q',1}}, \quad p \in [1, \infty], q \in \mathbb{N}, \quad q' := d + q - \frac{d}{p} \ge q. \tag{2}$$

$$\|\cdot\|_{\mathscr{C}^q} \le c\|\cdot\|_{W^{q',1}}, \quad q \in \mathbb{N}, q' := q + d. \tag{3}$$

Thus, it is sufficient to control the norms $\|\cdot\|_{W^{q,1}}$ for $q \in \mathbb{N}$, in order to control all the norms $\|\cdot\|_{W^{q,p}}$ for $q \in \mathbb{N}$ and $p \in [1, \infty]$, and all the norms $\|\cdot\|_{\mathscr{C}^q}$ for $q \in \mathbb{N}$. To achieve this control, it is convenient to work in Fourier modes. We define

$$\|f\|_{N^q} := \sup_{\|\vec{t}\| \ge 1} \|\vec{t}\|^q |\hat{f}(\vec{t})|, \quad q \in \mathbb{N}.$$

By the Hausdorff–Young inequality [20], Theorem 7.1.13, the Fourier isomorphism theorem [20], Theorem 7.1.11, the continuity of the Fourier transform on L^p-spaces [20], Theorem 7.9.3, and (2), we have

$$\|\cdot\|_{W^{q,1}} \le c\|\cdot\|_{N^{q+1}} \le c'\|\cdot\|_{W^{q+1,1}}. \tag{4}$$

THEOREM 8. *Let $(\vec{F}_n)$ be as in Theorem 4. Then, denoting by f_n the density of $\vec{F}_n$ and by φ the density of the standard Gaussian distribution on $\mathbb{R}^d$, for every $p \in [1, \infty]$ and $q \ge 0$, we have that, for n large enough, $f_n \in W^{q,p}(\mathbb{R}^d)$ and*

$$f_n \xrightarrow[n \to \infty]{W^{q,p}(\mathbb{R}^d)} \varphi. \tag{5}$$

Let us state some immediate consequences and remarks:

(a) In particular, we obtain that $f_n \to \varphi$ in $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty]$. This convergence is already new, except for $p = 1$, which is simply total variation. By (3) we also deduce that, for n large enough, $f_n \in \mathscr{C}^q(\mathbb{R}^d)$ and $\| f_n \to \varphi\|_{\mathscr{C}^q}$.

(b) By (4) Theorem 8 also gives estimates on characteristic functions of vectors in Wiener chaoses close, in law, to a Gaussian vector, which may be advantageous in some circumstances: for every $m \in \mathbb{N}^*$ and $q \in \mathbb{N}$, there exist $\delta = \delta_{q,m} > 0$ and $C = C_{q,m} > 0$ such that, for every $\vec{F} \in \mathcal{W}_m$, we have

$$d_{\mathrm{FM}}(\vec{F}, \mathcal{N}(0, I_d)) \leq \delta \Rightarrow \sup_{\|\vec{t}\| \geq 1} \|\vec{t}\|^q \big|\mathbf{E}\big[e^{i\vec{t}\cdot\vec{F}}\big]\big| \leq C. \tag{6}$$

(c) We could, in fact, be more precise and estimate the rate of convergence. For example, Theorem 4, together with [22], Theorem 4.4, yields for univariate random variables that, for every $p \in \mathbb{N}$, there exist C_p and $\alpha_p > 0$ such that, for every $F \in \mathcal{W}_m$ with density f,

$$\mathbf{E}[F^4] - 3 < \alpha_p \Rightarrow \Big[f \in \mathscr{C}^p \text{ and } \sup_{x \in \mathbb{R}} \|f - \varphi\|_{\mathscr{C}^p} \leq C_p\, \mathbf{E}[F^4 - 3]^{1/2}\Big]. \tag{7}$$

This estimate could be generalized to random vectors. It would be interesting and useful to derive an explicit expression for the quantities α_p and C_p. This rather demanding task falls beyond the scope of this article and could be explored in further contributions.

Let us now show how modulo, a well-known result in Malliavin calculus, recalled in Section 3.2.2, Theorem 4 implies Theorem 8. Similarly, Theorem 3 implies Theorems 1 and 2.

PROOF OF THEOREMS 1, 2 AND 8. By (2), (3) and (4), it is sufficient to show that

$$\|f_n - \varphi\|_{N^q} \to 0, \quad q \in \mathbb{N}.$$

The conclusion of Theorem 4, that is, (1), implies that

$$\limsup_{n \to \infty} \|f_n\|_{W^{q,p}} < \infty, \quad p \in [1, \infty], \, q \in \mathbb{N}. \tag{8}$$

For details on this fairly standard result, see Lemma 23 below. By (4), (8) yields

$$\limsup_{n \to \infty} \|f_n\|_{N^q} < \infty, \quad q \in \mathbb{N}. \tag{9}$$

Fix $\varepsilon > 0$, and let $A = \varepsilon^{-1}$. By the convergence in law, we find that

$$\sup_{1 \leq \|\vec{t}\| \leq A} \|\vec{t}\| \big\|\hat{f}_n(t) - \hat{\varphi}(t)\big\| \to 0.$$

On the other hand, we have

$$\sup_{\|\vec{t}\| \geq A} \|\vec{t}\|^q \big\|\hat{f}_n(t) - \hat{f}(t)\big\| \leq \varepsilon^q \|f_n - \varphi\|_{N^{2q}}.$$

Since by (9) and smoothness of φ,

$$c := \limsup_{n \to \infty} \|f_n - \varphi\|_{N^{2q}} < \infty,$$

we find that

$$\limsup_{n \to \infty} \sup_{\|\vec{t}\| \geq A} \|\vec{t}\|^q \big\|\hat{f}_n(t) - \hat{f}(t)\big\| \leq c\varepsilon^q.$$

This concludes the proof. □

1.3.2. *Regularization on sum of chaoses.* The results partially extend to random variables in a finite sum of Wiener chaoses. We let $\mathcal{W}_{\le m} := \bigoplus_{k=0}^{m} \mathcal{W}_k$. We also denote by J_k the projection on the kth Wiener chaos.

Theorem 1 does not hold on $\mathcal{W}_{\le m}$. Indeed, if $F_n = (n+1)^{-1}G^2 + G$ where G belongs to the first chaos of the underlying Gaussian field, then $(F_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}_{\le 2}$ converging in law to the standard Gaussian. Nonetheless, $\Gamma[F_n, F_n] = (2(n+1)^{-1}G + 1)^2$. It follows that $\Gamma[F_n, F_n]^{-1}$ is never integrable. Additionally, by direct computations the density of F_n is never continuous. Nevertheless, we obtain regularization results for sequences in $\mathcal{W}_{\le m}$ under some assumptions on the projection over the largest Wiener chaos $\mathcal{W}_m$.

THEOREM 9. *Fix $m \in \mathbb{N}^*$ and a sequence $(F_n) \subset \mathcal{W}_{\le m}$. We assume that*:

(i) $\mathsf{J}_m(F_n) \to \mathcal{N}(0, 1)$ *in law.*
(ii) $(F_n)_{n\ge 1}$ *is bounded in* L^2.

Then for every $q \ge 1$, there exist an integer N and a constant C such that

$$\mathbf{E}\big[\Gamma[F_n, F_n]^{-q}\big] \le C, \quad n \ge N.$$

Let us make some comments on this theorem:

- The main assumption concerns only the projection of F_n on $\mathcal{W}_m$. The projection on the other Wiener chaoses only need to be bounded.
- The conclusion implies that the density of F_n regularizes as n tends to $+\infty$. At this level of generality, $(F_n)_n$ does not converge in law. It is not possible to talk about smooth convergence of the densities. Nevertheless, by the same argument as in the proof of Theorem 8, our result implies that all the limits in law of a subsequence of (F_n) have a smooth density and that the subsequence of the densities converges smoothly.

Concerning smooth normal convergence, we state the following corollary. We write $\mathsf{D}^2 F$ for the *Malliavin Hessian* of F, and for an Hilbert–Schmidt operator A, we write $\rho(\mathsf{A})$ for its spectral radius.

COROLLARY 10. *Consider a sequence $(F_n) \subset \mathcal{W}_{\le m}$ and the sequence of associated densities (f_n). Assume any of the three following situations hold*:

(a) *$F_n - \mathsf{J}_m F_n \to 0$ in L^2 and (F_n) converge in law to a standard Gaussian.*
(b) *For every $k = 0, \dots, m$, $(\mathsf{J}_k F_n)$ converges in law to a Gaussian measure, possibly degenerate, except for $k = m$.*
(c) *(F_n) is bounded in L^2, $\rho(\mathsf{D}^2 F_n) \to 0$ in L^2, and* $\liminf \mathbf{Var}[\mathsf{J}_m F_n] > 0$.

Then we have superconvergence *to a Gaussian density*,

$$f_n \xrightarrow[n\to\infty]{W^{q,p}(\mathbb{R})} \varphi, \quad q \ge 0, p \in [1, \infty].$$

PROOF. The proofs are immediate but for (c). Since for $l \in \{1, \dots, m\}$,

$$\prod_{i=1, i\ne l}^{m} \left(\mathsf{L}^{-1} + \frac{1}{i}\right) = \left[\prod_{i=1, i\ne l}^{m} \left(\frac{1}{k} - \frac{1}{l}\right)\right] \mathsf{J}_l,$$

by Meyer's inequalities (see, for instance [37], equation (3.18)), we find that $\rho(\mathsf{D}^2 \mathsf{J}_l F_n) \to 0$ in L^2 and thus also in L^4 by equivalence of L^p norms on Wiener chaos (see Section 3.1.3 below). Since $F_n \in \mathcal{W}_{\le m}$, $\Gamma(F_n, F_n) \in \mathcal{W}_{\le m'}$ for $m' = 2(m-1)$ (see [36], Proposition 2.7.4

and Theorem 2.7.10). By equivalence of L^p norms on Wiener chaos and by [41], Proposition 1.2.2, we find that

$$\mathbf{E}\big[\|\mathsf{D}F_n\|^4\big]^{1/4} = \mathbf{E}\big[\Gamma(F_n, F_n)^2\big]^{1/4} \le c\,\mathbf{E}\big[\Gamma(F_n, F_n)\big]^{1/2} \le c'\,\mathbf{E}\big[F_n^2\big]^{1/2}.$$

Thus, we are in the setting of the second order Poincaré inequality [13, 37], and each of the $(\mathsf{J}_k F_n)$ converges in law to Gaussian, possibly degenerate but for $k = m$ in view of our assumption. We conclude by (b). □

Actually, from the proof of Theorem 4, one could obtain a vector-valued version of our theorem for sum of chaoses.

THEOREM 11. *Let $d \in \mathbb{N}^*$ and $m_1, \dots, m_d \in \mathbb{N}^*$. Consider a sequence $(\vec{F}_n) \subset \prod_i \mathcal{W}_{\le m_i}$. Assume that*:

(i) $(\mathsf{J}_{m_1} F_{n,1}, \dots, \mathsf{J}_{m_d} F_{n,d}) \to \mathcal{N}(0, I_d)$ *in law.*

(ii) $(\vec{F}_n)_{n\ge 1}$ *is bounded in* L^2.

Then for every $q \ge 1$, there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$\mathbf{E}\big[\det \Gamma(\vec{F}_n)^{-q}\big] \le C, \quad n \ge N.$$

1.4. *Scheme of the proof of the main results.* Following Malliavin's idea, Lemma 23, the core of the proof is to establish control of negative moments of $\Gamma[F_n, F_n] = \|\mathsf{D}F_n\|^2$ for a sequence $(F_n) \subset \mathcal{W}_m$ asymptotically normal, that is, Theorem 3. Actually, it is sufficient to prove the claim for functionals, depending on finitely many independent Gaussian variables $(N_1, \dots, N_K)$ with K arbitrarily large and all the estimates being independent of K, as explained in Section 3.1.4. In this setting we prove Theorem 3 by induction on m. The main steps of the proof are as follows.

Extending the statement to vectors. Through a discretization procedure, we show, in Corollary 32, that the induction hypothesis, that is, Theorem 3 for $m-1$, implies its vectorial version, that is, Theorem 4, restricted to $(\vec{F}_n) \subset \mathcal{W}_{m-1}^d$.

Negative moments for the derivative and spectral remainders of the Hessian. Our key observation relates the negative moments of $\Gamma[F_n, F_n]$ to spectral quantities associated to the Malliavin Hessian $\mathsf{D}^2 F_n$. Namely, for all $q \in \mathbb{N}^*$, we introduce the spectral quantities

$$\mathcal{R}_q\big(\mathsf{D}^2 F_n\big) := \sum_{i_1 \ne \cdots \ne i_q} \lambda_{i_1}^2 \dots \lambda_{i_q}^2,$$

where (λ_i) is the spectrum of the random matrix $\mathsf{D}^2 F_n$. Then in Proposition 30 we show that

$$\mathbf{E}\big[\Gamma[F_n, F_n]^{-q}\big] \le c\,\mathbf{E}\big[\mathcal{R}_{q'}\big(\mathsf{D}^2 F_n\big)^{-1/2}\big],$$

where q' depends only on q. Whenever $m = 2$, that is, on the second Wiener chaos, $\mathsf{D}^2 F_n$ is a deterministic matrix, and the above inequality follows from a diagonalization argument together with an explicit computation of the Fourier transform of a chi-squared distribution. This step is completed in Proposition 27. For the case of higher degree, we use a new decoupling idea based on taking Malliavin derivatives in the direction of Gaussian random variables independent of the underlying field.

Compressing the Hessian. Actually, we do not directly derive estimates on $\mathcal{R}_q(\mathsf{D}^2 F_n)$ but rather control a compressed matrix. To do so, we generalize, using singular values, the notion of spectral remainders to rectangular matrices (see (19)). Then we show in Lemmas 39 and 40 that the set of matrices X of size $K \times q$ such that negative moments of $\mathcal{R}_q(\mathsf{D}^2 F_n \cdot X)$ controls those of $\mathcal{R}_q(\mathsf{D}^2 F_n)$ is *large*, in a measure-theoretical sense. The idea is to take X, a Gaussian random matrix independent of the underlying field, and show that this control happens with high probability.

Control of the compressed Hessian. We connect the compressed Hessian with the Malliavin matrix of an intermediary random vector living in a Wiener chaos of degree $m-1$. Namely, define

$$\mathsf{D}_{\vec{x}} F_n := \sum_{k=1}^{K} \frac{\partial F_n}{\partial N_k} x_k \in \mathcal{W}_{m-1}, \quad \vec{x} = (x_1, \dots, x_K) \in \mathbb{R}^K;$$
$$\mathsf{D}_X F_n := (\mathsf{D}_{\vec{x}_1} F_n, \dots, \mathsf{D}_{\vec{x}_d} F_n) \in \mathcal{W}_{m-1}^d, \quad X = (\vec{x}_1, \dots, \vec{x}_d) \in \mathbb{R}^{K \times d}.$$

We show that $\mathcal{R}_q(\mathsf{D}^2 F_n \cdot X) = \det(\Gamma(\mathsf{D}_X F_n))$. Then in Lemma 38 we exhibit a set of matrices X, with large Gaussian measure, such that the law of $\mathsf{D}_X F_n$ is close to a Gaussian. To do so, with respect to the enlarged Gaussian field (N_k), (X_{ij}), we have that $\mathsf{D}_X F_n \in \mathcal{W}_m^d$, and we conclude thanks to well-known results, linking asymptotic normality on Wiener chaoses and convergence of the norm of the Malliavin derivative to a constant, that are recalled in Section 3.2.3.

Conclusion. Since the two sets of matrices constructed in the previous steps have large Gaussian measure, say greater than $2/3$, they have nonempty intersection. Therefore, our construction yields a matrix X such that the two conditions hold simultaneously. Since $\mathcal{R}_q(\mathsf{D}^2 F_n \cdot X) = \det(\Gamma(\mathsf{D}_X F_n))$ and $(\mathsf{D}_X F_n) \subset \mathcal{W}_m^{d-1}$, by the vectorial version of the induction hypothesis, we conclude the induction step using that $(\mathsf{D}_X F_n)$ is asymptotically normal.

Remark on the proof: The importance of Gaussian variables. Evaluating directional derivatives in independent Gaussian variables plays a decisive role in several steps of the proof. In this short paragraph, we would like to ease the reader's acclimation to this new paradigm. The usual Malliavin derivative of a random variable F is defined in the direction of $h \in \mathfrak{H}$, where $\mathfrak{H}$ is an abstract separable Hilbert space. Due to the isomorphisms between separable Hilbert spaces, the literature has maintained that the choice of $\mathfrak{H}$ is inconsequential. The present study, together with our companion paper [17] where we use similar ideas in a non-Gaussian setting, puts forward a preferred choice for $\mathfrak{H}$: a Gaussian space, independent of the underlying Gaussian field. Such choice guarantees that the Malliavin derivative is an element of an enlarged Wiener space, as defined in (12), and allows us to put into action all the fine results regarding the Wiener space. Historically, we trace back this idea to [9], where Bouleau chooses $\mathfrak{H}$ to be a copy of the underlying L^2 space.

2. Applications.

Our result expresses a broad and versatile phenomenon. Numerous statements establish normal convergence for polynomial functionals of a Gaussian field. Our conclusions potentially comprehend all these situations. We illustrate the flexibility and the breadth of our analysis with applications coming from different fields, without trying to be exhaustive or stating optimal results.

2.1. *Small ball estimates for multilinear Gaussian polynomials.*

The celebrated inequality of Carbery and Wright [11] states that, for $(G_1, \dots, G_n)$ a standard Gaussian vector and P a polynomial of degree d such that

$$\mathbf{E}\big[\big|P(G_1, \dots, G_n)\big|\big] = 1,$$

we have

$$\mathbf{P}\big[\big|P(G_1, \dots, G_n)\big| \le \epsilon\big] \le c_d \epsilon^{\frac{1}{d}},$$

where c_d depends on d only and is independent from n. Applied to multilinear homogeneous sums, this inequality plays a crucial role in the seminal contribution [31]. They obtain quantitative invariance principles in various convergence metrics, and for the roughest metrics, the resulting bounds may depend on d through the exponents of the maximal influence. A multilinear homogeneous sum evaluated in a standard Gaussian vector being an archetypal example of Wiener chaos, our Theorem 8 applies. Thus, provided $d_{\mathrm{FM}}(P(G_1,\dots,G_n),\mathcal{N}(0,1))$ is small enough, the random variable $P(G_1,\dots,G_n)$ has a bounded density. This implies that

$$\mathbf{P}\big[\big|P(G_1,\dots,G_n)\big| \leq \epsilon\big] \leq c_d \epsilon$$

for an another constant c_d. This considerably improves the exponent on ϵ.

2.2. *Smooth convergence in Breuer–Major theorem.* Consider $(X_n)_{n\in\mathbb{Z}}$ a stationary sequence of centered and normalized Gaussian variables and $f \in L^2(\gamma)$, where $\gamma := \mathcal{N}(0,1)$. Breuer and Major [10] give sufficient conditions for the asymptotic normality of $Z_n := n^{-1/2}\sum_{k=1}^n f(X_k)$. Define the Hermite rank of f as the smallest integer s such that the projection of f on the sth Hermite polynomial H_s is nonzero. [10] proves that if the correlation function $\rho(k) := \mathbf{E}[X_0 X_k]$ belongs to $\ell^s(\mathbb{N})$, then (Z_n) converges in law to a Gaussian distribution. In particular, whenever $\rho \in \ell^1(\mathbb{N})$, (Z_n) converges in law to a Gaussian for any $f \in L^2(\gamma)$. When f is a polynomial, Hu, Nualart, Tindel and Xu [23] give conditions to ensure $\mathscr{C}^\infty$-convergence of the densities, in terms of logarithmic integrability of the spectral density. They use their conditions to control the negative moments of the Malliavin derivative of (Z_n). Since our results provide such controls as soon as we have normal convergence, we obtain that the $\mathscr{C}^\infty$-convergence holds without any additional assumption.

THEOREM 12. *Let (X_n) be a stationary normalized Gaussian sequence and P a polynomial. Assume that the correlation function ρ belongs to $\ell^s(\mathbb{N})$, where s is the Hermite rank of P. Then the density of $Z_n = n^{-1/2}\sum_{k=1}^n P(X_k)$ converges to a Gaussian density in $W^{q,p}(\mathbb{R})$ for every $q \geq 0$ and $p \in [1,\infty]$.*

PROOF. Seeing the variables (X_k) as elements of a Gaussian field, (Z_n) belongs to $\mathcal{W}_{\leq m}$, where $m = \deg(P)$. Writing $P = \sum_{i=s}^m c_i H_i$, the projections of Z_n on $\mathcal{W}_i$ are given by

$$\mathsf{J}_i(Z_n) = \frac{c_i}{\sqrt{n}}\sum_{k=1}^n H_i(X_k), \quad i = s,\dots,m.$$

By [10] each projection $\mathsf{J}_i(Z_n)$ converges in law to a Gaussian variable, nondegenerate for $i = m$ since $c_m \neq 0$. The result follows from Corollary 10 (b). □

2.3. *Normal convergence in entropy and Fisher information.* Let us recall some notions from information theory. Let φ be the density of the standard Gaussian distribution on $\mathbb{R}^d$. Let $\vec{F}$ be a random vector of $\mathbb{R}^d$ with density f. The *relative entropy* of $\vec{F}$ with respect to $\mathcal{N}(0, I_d)$ is

$$\mathbf{Ent}[\vec{F}] := \mathbf{E}\left[\log\frac{f(\vec{F})}{\varphi(\vec{F})}\right] = \mathbf{E}[\log f(\vec{F})] + \frac{1}{2}\mathbf{E}[\|\vec{F}\|^2] + \frac{d}{2}\log 2\pi,$$

while its *relative Fisher information* is

$$\mathbf{I}[\vec{F}] := \mathbf{E}\left[\left\|\vec{\nabla}\log\frac{f(\vec{F})}{\varphi(\vec{F})}\right\|^2\right] = \int_{\mathbb{R}^d}\frac{\|\vec{\nabla}f(x)\|^2}{f(x)}\,\mathrm{d}x - \mathbf{E}[\|\vec{F}\|^2].$$

The total variation distance, the relative entropy and the relative Fisher information are related through Pinsker's inequality, and the log-Sobolev inequality [15]

$$d_{\mathrm{TV}}\big(\vec{F},\mathcal{N}(0,I_d)\big)^2 \leq \frac{1}{2}\mathbf{Ent}[\vec{F}] \leq \frac{1}{4}\mathbf{I}[\vec{F}].$$

Thus, convergence in Fisher information is an improvement to the convergence in entropy, which is itself an improvement to the convergence in total variation. Finally, we define the multivariate *score function* of $\vec{F}$

$$\vec{\rho} = (\rho_1, \dots, \rho_d) := \vec{\nabla} \log f.$$

In this way

$$\mathbf{I}[\vec{F}] = \mathbf{E}\big[\|\vec{\rho}(\vec{F})\|^2 - \|\vec{F}\|^2\big] = \sum_{i=1}^{d} \mathbf{E}\big[\rho_i(\vec{F})^2 - F_i^2\big].$$

The following integration by parts characterises the score function:

$$\mathbf{E}\big[\partial_i \Phi(\vec{F})\big] = \mathbf{E}\big[\rho_i(\vec{F})\Phi(\vec{F})\big], \quad \Phi \in \mathscr{C}_c^1(\mathbb{R}^d), i = 1, \dots, d. \tag{10}$$

Consider a sequence of isotropic vectors $(\vec{F}_n) \subset \mathcal{W}_m^d$ converging in law to $\vec{N}$, the standard Gaussian vector on $\mathbb{R}^d$. We recall that isotropic means that the covariance matrix of $\vec{F}_n$ is I_d for all $n \in \mathbb{N}$. Let f_n be the density of $\vec{F}_n$. By [39] we have that $\mathbf{Ent}[\vec{F}_n] \to 0$. More precisely, they show the bound,

$$\mathbf{Ent}[\vec{F}_n] \le O\big(\Delta_n |\log \Delta_n|\big), \quad \Delta_n := \mathbf{E}\big[\|\vec{F}_n\|^4 - \|\vec{N}\|^4\big].$$

[39] actually provides an analogous bound for nonisotropic random vectors. We focus on the isotropic case for simplicity. The general case can be obtained by multiplying all the $\vec{F}_n$ by the square root of the inverse of their covariance matrix. This bound is suboptimal since by [1, 6, 25], in the case of sums of i.i.d., centred and normalized random variables $S_n = n^{-1/2} \sum_{k=1}^{n} X_k$, we have

$$\mathbf{I}[S_n] \le O\big(n^{-1}\big).$$

Our findings allow us to improve upon the results of [39] and to provide an optimal rate of convergence in entropy on Wiener chaoses. Actually, we obtain directly an optimal rate of convergence in Fisher information.

THEOREM 13. *Fix $d \in \mathbb{N}^*$, $m_1, \dots, m_d \in \mathbb{N}^*$ and a sequence of isotropic random vectors $(\vec{F}_n)_{n\in\mathbb{N}} \subset \prod_i \mathcal{W}_{m_i}$. Assume that $(\vec{F}_n)$ converges in law to the standard d-dimensional Gaussian distribution $\vec{N}$. Then there exists a constant C such that, for n in $\mathbb{N}$ large enough,*

$$\mathbf{Ent}[\vec{F}_n] \le \frac{1}{2}\mathbf{I}[\vec{F}_n] \le C\Delta_n, \quad \Delta_n := \mathbf{E}\big[\|\vec{F}_n\|^4 - \|\vec{N}\|^4\big].$$

Actually, from this result we can obtain a uniform bound for the relative entropy in the case $d = 1$.

COROLLARY 14. *Fix $m \in \mathbb{N}^*$. There exists a constant $C = C_m$ such for any $F \in \mathcal{W}_m$ with unit variance, we have*

$$\mathbf{Ent}[F] \le C\,\mathbf{E}\big[F^4 - 3\big].$$

REMARK 15. Thanks to Pinsker inequality, Corollary 14 implies the celebrated inequality [35],

$$d_{\mathrm{TV}}\big(F, \mathcal{N}(0,1)\big)^2 \le C\,\mathbf{E}\big[F^4 - 3\big], \quad F \in \mathcal{W}_m \;\; \mathbf{E}\,F^2 = 1.$$

However, in the inequality of [35], the constant C does not depend on the order of the chaos m. For instance, [14] gives $C = 1/3$.

PROOF OF COROLLARY 14. Since $\mathbf{E}\,\Gamma(F, F) = m$, arguing as in [34], Proposition 4.2, one can find $p = p_m > 1$ such that the L^p-norm of the density of F is uniformly bounded; thus, there exists a constant $C > 0$ depending only on the m such that $\mathbf{Ent}[F] \le C$, uniformly in $F \in \mathcal{W}_m$ with unit variance. Theorem 13 gives $\delta > 0$ and $C' > 0$ such that for any such F, if $\Delta := \mathbf{E}[F^4 - 3] < \delta$, then $\mathbf{Ent}[F] \le C'\Delta$. These two observations implies the claim. □

REMARK 16. Let us comment on the extension of Corollary 14 to the multivariate case. Following [34], a uniform bound $\mathbf{E}[\det \Gamma(\vec{F})] \ge \beta$ for some $\beta > 0$ yields a uniform upper bound for $\mathbf{Ent}[\vec{F}]$. If we assume such bound, we could implement the same strategy as above. We stress that the mere assumptions $\vec{F} \in \prod_i \mathcal{W}_{m_i}$ isotropic does not imply $\mathbf{E}[\det \Gamma(\vec{F})] > 0$.

The proof of Theorem 13 relies on two lemmas. The first lemma is a consequence of our main result.

LEMMA 17. *Let $d \in \mathbb{N}^*$ and $m_1, \dots, m_d \in \mathbb{N}^*$. Consider a sequence of isotropic vectors $(\vec{F}_n) \subset \prod_i \mathcal{W}_{m_i}$. Let $\vec{N} \sim \mathcal{N}(0, I_d)$. Assume that*

$$\vec{F}_n \xrightarrow[n\to\infty]{\text{Law}} \vec{N}.$$

Define

$$W_n := (W_n^{ij})_{1\le i,j\le d} := \Gamma(\vec{F}_n)^{-1};$$

$$W := (W^{ij})_{1\le i,j\le d} := \operatorname{diag}(m_1^{-1}, \dots, m_d^{-1}).$$

Then for every $p \ge 1$, there exist $C > 0$ and $N > 0$ such that, for every $n \ge N$ and every $1 \le i, j \le d$,

$$\|W_n^{ij} - W^{ij}\|_{L^p} + \|\Gamma[W_n^{ij}, W_n^{ij}]\|_{L^p} \le C\Delta_n^{1/2}, \quad \Delta_n := \mathbf{E}[\|\vec{F}_n\|^4 - \|\vec{N}\|^4].$$

PROOF. We fix a matrix norm $\|\cdot\|$, and $p \ge 1$. If M is a random matrix, we write

$$\|M\|_{L^p} := \mathbf{E}[\|M\|^p]^{\frac{1}{p}}.$$

As a consequence of Theorem 4, there exists a constant $C > 0$ such that, for n large enough,

$$\|\Gamma(\vec{F}_n)^{-1}\|_{L^{2p}} \le C.$$

By hypercontractivity (13) and then [40], Proof of Theorem 4.2, there exists $C > 0$ such that, for any $n \ge 1$,

$$\|\Gamma(\vec{F}_n) - D\|_{L^{2p}} \le C_0\|\Gamma(\vec{F}_n) - D\|_{L^2} \le C\Delta_n^{1/2},$$

where $D := \operatorname{diag}(m_1, \dots, m_d)$. Using that

$$\Gamma(\vec{F}_n)^{-1} - D^{-1} = \Gamma(\vec{F}_n)^{-1}(D - \Gamma(\vec{F}_n))D^{-1},$$

we deduce by continuity of the matrix product and the Cauchy–Schwarz inequality that there exists $C > 0$ such that, for n large enough,

$$\|\Gamma(\vec{F}_n)^{-1} - D^{-1}\|_{L^p} \le C\Delta_n^{1/2}.$$

Equivalently, $\|W_n - W\|_{L^p} \le C\Delta_n^{1/2}$; thus, $\sup_{i,j} \|W_n^{ij} - W^{ij}\|_{L^p} \le C\Delta_n^{1/2}$ with possibly another constant $C > 0$. This shows the inequality for the first term in the left-hand side.

For the second term, by Cramer formula

$$W_n^{ij} - W^{ij} = \frac{Z_n^{ij}}{\det(\Gamma(\vec{F}_n))},$$

where Z_n^{ij} is polynomial in the entries of $\Gamma(\vec{F}_n)$. We have $\|Z_n^{ij}\|_{L^{2p}} \leq C\Delta_n^{1/2}$, and since Z_n^{ij} is polynomial, we deduce, by integration by parts, the bound $\|\Gamma[Z_n^{ij}, Z_n^{ij}]\|_{L^{2p}} \leq C\Delta_n^{1/2}$. These two bounds and the bound on negative moments of $\det(\Gamma(\vec{F}_n))$ give the bound $\|\Gamma[Z_n^{ij}, Z_n^{ij}]\|_{L^p} \leq C\Delta_n^{1/2}$ for n large enough. □

The second lemma provides, on Wiener chaoses, an explicit formula for the score function through integration by parts with Malliavin operators.

LEMMA 18. *Let $d \in \mathbb{N}^*$, $m_1, \dots, m_d \in \mathbb{N}^*$. Take $\vec{F} = (F_1, \dots, F_d) \in \prod_i \mathcal{W}_{m_i}$. Define $W = (W_{ij})_{1\leq i,j\leq d} := \Gamma(F)^{-1}$. Then the score function $\vec{\rho} = (\rho_1, \dots, \rho_d)$ of $\vec{F}$ is given by*

$$\rho_i(\vec{F}) := \sum_{j=1}^d \mathbf{E}\big[m_j F_j W_{ij} - \Gamma[F_j, W_{ij}] | \vec{F}\big].$$

PROOF. Fix $\Phi \in \mathscr{C}_c^1(\mathbb{R}^d)$. By the chain rule for Γ,

$$\Gamma\big[\Phi(\vec{F}), F_i\big] = \sum_{j=1}^d \partial_j \Phi(\vec{F}) \Gamma[F_i, F_j], \quad i = 1, \dots, d.$$

In matrix notation,

$$\begin{pmatrix} \Gamma[\Phi(\vec{F}), F_1] \\ \vdots \\ \Gamma[\Phi(\vec{F}), F_d] \end{pmatrix} = \Gamma(\vec{F}) \begin{pmatrix} \partial_1 \Phi(\vec{F}) \\ \vdots \\ \partial_d \Phi(\vec{F}) \end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} \partial_1 \Phi(\vec{F}) \\ \vdots \\ \partial_d \Phi(\vec{F}) \end{pmatrix} = W \begin{pmatrix} \Gamma[\Phi(\vec{F}), F_1] \\ \vdots \\ \Gamma[\Phi(\vec{F}), F_d] \end{pmatrix}.$$

Thus, we find

$$\partial_i \Phi(\vec{F}) = \sum_{j=1}^d W_{ij} \Gamma\big[\Phi(\vec{F}), F_j\big], \quad i = 1, \dots, d.$$

By integration by parts, we deduce

$$\begin{aligned} \mathbf{E}\big[\partial_i \Phi(\vec{F})\big] &= \sum_{j=1}^d \mathbf{E}\big[W_{ij} \Gamma\big[\Phi(\vec{F}), F_j\big]\big] \\ &= \sum_{j=1}^d \mathbf{E}\big[W_{ij} \Phi(\vec{F}) m_j F_j\big] - \mathbf{E}\big[\Gamma[F_j, W_{ij}] \Phi(\vec{F})\big] \\ &= \sum_{j=1}^d \mathbf{E}\big[\big(m_j F_j W_{ij} - \Gamma[F_j, W_{ij}]\big) \Phi(\vec{F})\big]. \end{aligned}$$

The result follows in view of (10). □

PROOF OF THEOREM 13. For the sake of conciseness, we drop the dependence in n in this proof. By Lemma 18 we have

$$\rho_i(\vec{F}) - F_i = \mathbf{E}[Z_i | \vec{F}],$$

where

$$Z_i = (m_i - W_{ii}^{-1}) W_{ii} F_i + \sum_{j \neq i} m_j F_j W_{ij} - \Gamma[F_j, W_{ij}].$$

By Lemma 17 we find that

$$\|Z_i\| \leq C \Delta^{1/2}.$$

In this case we conclude by the triangle inequality. □

2.4. *Regularization of spectral moments of random matrices.* In linear algebra many quantities of interest, such as moments of the spectral measure, are polynomials in the entries of the matrix. Thus, the theory of random matrices provides another context where our results naturally apply.

Let $n \in \mathbb{N}^*$. The *Gaussian Orthogonal Ensemble* $\mathrm{GOE}(n)$ is the probability distribution on the set of $n \times n$ symmetric matrices with density with respect to the Lebesgue measure is proportional to $\exp(-n \operatorname{Tr}(A^2)/4)$. Equivalently, a random $n \times n$ symmetric matrix $A_n \sim \mathrm{GOE}(n)$ if and only if the entries of A_n above the diagonal are independent Gaussian, with variance n^{-1} out of the diagonal and with variance $2n^{-1}$ on the diagonal. Following the famous semicircle law [46], when properly rescaled, the moments of the spectral measure of A_n converges to the respective moments of the semicircle law,

$$\frac{1}{n} \operatorname{Tr} A_n^p \xrightarrow[n\to\infty]{a.s.} c_p := \frac{1}{2\pi} \int_{-2}^{2} x^p (4 - |x|^2)^{1/2} \, \mathrm{d}x.$$

Moreover, by [24] the normalized fluctuations $\operatorname{Tr}(A_n^p) - nc_p$ converge in distribution to a Gaussian limit $\mathcal{N}(0, \sigma_p^2)$ for some $\sigma_p \neq 0$.

We stress that both Wigner [46] and Johansson [24] results are actually available for symmetric random matrices with entries possibly non-Gaussian. In the case of Gaussian entries, our results improve the mode of convergence of the fluctuations.

THEOREM 19. *Let $A_n \sim \mathrm{GOE}(n)$ for each $n \in \mathbb{N}^*$, and let $p \geq 1$. Then the sequence of densities of $\operatorname{Tr}(A_n^p) - nc_p$ converges to a Gaussian density in $W^{q,r}(\mathbb{R})$ for every $q \geq 0$ and $r \in [1, \infty]$.*

PROOF. For the convergence of densities in Sobolev spaces, we use Corollary 10 (c). Consider the Gaussian field on $\mathbb{N}^* \times \mathbb{N}^*$ of independent standard Gaussian $(G_{i,j})$. Write $F_n := \operatorname{Tr} A_n^p - nc_p$. We have that

$$\operatorname{Tr} A_n^p = \frac{1}{n^{\frac{p}{2}}} \sum_{1 \leq i_1, \ldots, i_p \leq n} G_{i_1,i_2} G_{i_2,i_3} \cdots G_{i_{p-1},i_p} G_{i_p,i_1}.$$

Thus, $F_n \in \mathcal{W}_{\leq p}$. [13] shows that $\rho(\mathsf{D}^2 F_n) \to 0$. Moreover, (F_n) is bounded in L^2. We are left to verify that $\mathbf{Var}[\mathsf{J}_p F_n] \geq O(1)$. Let us write

$$\mathscr{I}_p := \{(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p : \{i_l, i_{l+1}\} \neq \{i_{l'}, i_{l'+1}\}, l = 1, \ldots, p\}.$$

In view of the explicit expression of F_n, we find that

$$\mathsf{J}_p F_n = \frac{1}{n^{p/2}} \sum_{(i_1,\ldots,i_p) \in \mathscr{I}_p} G_{i_1,i_2} G_{i_2,i_3} \cdots G_{i_{p-1},i_p} G_{i_p,i_1} + R_n.$$

In the expression above and in view of the definition of $\mathscr{I}_p$, all the random variables appearing in the sum are independent. The term R_n, whose explicit expression is irrelevant, consists in sums of degree p of products of Hermite polynomials evaluated in the G_{ij}'s, at least one of these polynomials being of degree strictly greater than 1. By independence of the G_{ij}'s, the first sum and R_n are uncorrelated. Thus,

$$\begin{aligned}\mathbf{Var}[\mathsf{J}_p F_n] &\geq \frac{1}{n^p}\mathbf{E}\Big[\Big(\sum_{(i_1,\dots,i_p)\in\mathscr{I}_p} G_{i_1,i_2}G_{i_2,i_3}\dots G_{i_{p-1},i_p}G_{i_p,i_1}\Big)^2\Big]\\ &\geq \frac{c_p}{n^p}\big|\{(i_1,\dots,i_p)\in\{1,\dots,n\}^p : i_1 < i_2 < \dots < i_p\}\big|\\ &= \frac{c_p}{n^p}\binom{n}{p} \geq O(1).\end{aligned}$$

Corollary 10 gives the announced convergence in $W^{q,r}(\mathbb{R})$ for every $q \leq 0$ and $r \in [1,\infty]$. □

REMARK 20. We stress that our approach is rather general and could be extended to a situation where the $(G_{i,j})$ have more general variances, that is, $c_1 n^{-1} \leq \mathbf{Var}[G_{i,j}] \leq c_2 n^{-1}$ for some constants $c_1, c_2 > 0$.

2.5. *Control of the inverse of strongly correlated Wishart-type matrices.* Fix n and $d \in \mathbb{N}^*$. Let B be a $n \times d$ matrix whose lines are independent random vectors of $\mathbb{R}^d$ with common distribution $\mathcal{N}(0,\Sigma)$. *Wishart matrices* are, in their classical sense, matrices of the form $A = {}^tBB$. We can see the lines of B_n as realizations of normal experiments and see $A_n := \frac{1}{n}{}^tB_nB_n$ as the empirical covariance matrix of the sample. When p is fixed and $n \to +\infty$, after renormalization the sequence of Wishart matrices converges to the actual covariance,

$$A_n \to \Sigma.$$

We consider a broad generalization of Wishart matrices. The lines of B_n are not necessarily independent, nor identically distributed, and we only assume the convergence property $A_n \to \Sigma$. We then obtain a good control on the inverse A_n^{-1}. Our general version allows for correlation in the sample and might be of interest in statistics.

THEOREM 21. *Fix n and $p \in \mathbb{N}^*$. Let $A_n = {}^tB_nB_n$, where B_n is a matrix $n \times p$ with entries in a Gaussian field. We assume that (A_n) converges in probability to a deterministic invertible matrix Σ. Then for every $q \geq 1$, there exist an integer N and a constant C such that*

$$\mathbf{E}\big[\det(A_n)^{-q}\big] \leq C, \quad n \geq N.$$

In particular, $A_n^{-1} \to \Sigma^{-1}$ in L^q for every $q \geq 1$.

PROOF. Consider a Gaussian vector $\vec{G} = (G_1,\dots,G_d) \sim \mathcal{N}(0, I_d)$. Without loss of generality, assume that the G_i's are elements of the underlying Gaussian field but independent of the entries of A_n. Let $\vec{F}_n := B_n\vec{G} \in \mathcal{W}_2^d$. Conditionally to B_n, $\vec{F}_n$ is a Gaussian vector with covariance matrix ${}^tB_nB_n = A_n$. Since, by assumption, $A_n \to \Sigma$ in probability, we deduce that $\vec{F}_n$ converges in distribution to $\mathcal{N}(0,\Sigma)$. By Theorem 4 and for n large enough,

$$\mathbf{E}\big[\det(\Gamma(\vec{F}_n))^{-q}\big] \leq c.$$

We have $F_i = \sum_k B_n[i,k]G_k$; thus, by bilinearity and independence of B_n and $\vec{G}$,

$$\begin{aligned}\Gamma(F_i,F_j) &= \sum_{k,l}\Gamma\big(B_n[k,i], G_k, B_n[l,j]G_j\big)\\ &= \sum_{k,l} G_kG_l\Gamma\big(B_n[k,i], B_n[l,j]\big) + \sum_{k,l} B_n[k,i]\Gamma(G_i,G_j)B_n[l,j].\end{aligned}$$

Defining M is the nonnegative random matrix, whose entries are given by

$$M_{ij} := \sum_{k,l} G_k \Gamma\big(B_n[i,k], B_n[j,l]\big) G_l,$$

we thus have, using that $\Gamma(\vec{G}) = I_d$,

$$\Gamma(\vec{F}_n) = {}^t B_n B_n + M = A_n + M.$$

This give $\Gamma(\vec{F}_n) \geq A_n$. The conclusion follows. □

REMARK 22.

(a) For simplicity, our statement is formulated for a matrix B_n with entries taking values in a Gaussian field. From the proof, we see that the conclusion of the theorem remains valid if we take the entries of B_n with values in a Wiener chaos $\mathcal{W}_m$.

(b) Whenever the lines of B_n are i.i.d., developing the determinant with the Cauchy–Binet formula and bounding it from below by a sum of positive independent terms yield a more direct proof.

(c) We rely on similar strategy to obtain explicit estimates in the proof of our main theorems.

3. Prolegomena on Wiener chaoses. In this section we provide the necessary definitions, notations, and preliminary lemmas required for the proof of the main theorem. In all the article, for any parameter α, C_α stands for a constant which only depends on α and whose value may possibly change from line to line. Nevertheless, for the sake of clarity, we generally do not track the dependence on the order of the chaoses, typically denoted by m.

3.1. *Succinct review on Wiener chaoses.*

3.1.1. *The Wiener space.* We let $\gamma := \mathcal{N}(0,1)$ be the standard Gaussian distribution on $\mathbb{R}$. We work on the following countable product of probability spaces, which we call *a Wiener space*:

$$(\Omega, \mathscr{F}, \mathbf{P}) := \big(\mathbb{R}, \mathscr{B}(\mathbb{R}), \gamma\big)^{\mathbb{N}}. \tag{11}$$

We define the *coordinate functions*

$$N_i := \begin{cases} \Omega & \longrightarrow \mathbb{R}, \\ (x_0, x_1, \dots) & \longmapsto x_i. \end{cases}$$

By construction, the N_i's are independent random variables on $(\Omega, \mathscr{F}, \mathbf{P})$, with common law the standard Gaussian distribution. We sometimes require countably many auxiliary independent standard Gaussian random variables, say (N_i'), independent of (N_i). In the same way, we build such a family as coordinates of auxiliary Wiener spaces, and one is left to work for instance on an enlarged Wiener space

$$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbf{P}}) := \big(\mathbb{R}, \mathscr{B}(\mathbb{R}), \gamma\big)^{\mathbb{N}} \times \big(\mathbb{R}, \mathscr{B}(\mathbb{R}), \gamma\big)^{\mathbb{N}}. \tag{12}$$

Since $\mathbb{N}^2$ is equipotent with $\mathbb{N}$, an enlarged Wiener space is actually a Wiener space. In particular, we typically do not explicitly refer to this enlarging construction.

3.1.2. *The Wiener chaoses.* The *Hermite polynomials*

$$H_k(x) := (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad k \in \mathbb{N}$$

form a Hilbert basis of $L^2(\gamma)$. For $m \in \mathbb{N}$, the *mth Wiener chaos* $\mathcal{W}_m$ is defined as the $L^2(\mathbf{P})$-closure of the linear span of functions of the form $\prod_{i=0}^{\infty} H_{k_i}$, where the k_i's satisfy $\sum_{i=0}^{\infty} k_i = m$. The above product is finite since $H_0(x) = 1$ and only finitely many integers $(k_i)_{i \geq 0}$ are nonzero. For $m = 0$, we find that $\mathcal{W}_0$ is the linear space of constant functions. Importantly, Wiener chaoses provide the orthogonal decomposition

$$L^2(\mathbf{P}) = \bigoplus_{m=0}^{\infty} \mathcal{W}_m.$$

We sometimes work in a finite sum of Wiener chaoses. Accordingly, let us define

$$\mathcal{W}_{\leq m} := \bigoplus_{k=0}^{m} \mathcal{W}_k.$$

3.1.3. *Hypercontractivity and equivalence of norms.* We often use that on $\mathcal{W}_{\leq m}$ all the $L^p(\mathbf{P})$-norms are equivalent. Namely, for all $m \in \mathbb{N}$ and $1 \leq p < q < \infty$, there exists $c = c_{m,p,q}$ such that

$$\|F\|_p \leq \|F\|_q \leq c\|F\|_p, \quad F \in \mathcal{W}_{\leq m}. \tag{13}$$

This fact is well known in the range $1 < p < q < \infty$ as a consequence of *hypercontractivity* estimates, for instance [36], Cor. 2.8.14. The equivalence of norms can then be extended to the case $p = 1$ with an interpolation argument that we recall now. Fix $p = 1$ and $q \in (p, \infty)$. Of course, we only need to show the last inequality in (13). Take $F \in \mathcal{W}_{\leq m}$. A celebrated interpolation inequality, which is a consequence of Hölder's inequality, states that

$$\|F\|_{p_\theta} \leq \|F\|_{p_0}^{1-\theta} \|F\|_{p_1}^{\theta}, \quad p_0,\ p_1 \in [1, \infty],\ \theta \in (0, 1),$$

where $\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. With $p_0 = p = 1$ and $p_1 = q + 1$, there exists $\theta \in (0, 1)$ such that $p_\theta = q$. Thus, we find, by hypercontractivity,

$$\|F\|_q \leq \|F\|_1^{1-\theta} \|F\|_{q+1}^{\theta} \leq c_{m,q,q+1} \|F\|_1^{1-\theta} \|F\|_q^{\theta}.$$

This gives the announced extension.

3.1.4. *Reduction to finitely generated Wiener chaoses.* For the sake of simplicity, we conveniently work with *finitely generated* Wiener chaoses $\mathcal{W}_m^{(0)}$ defined as the linear span of functions of the form $\prod_{i=0}^{\infty} H_{k_i}$ with the k_i's subject to $\sum_{i=0}^{\infty} k_i = m$. This simplification avoids the use of infinite dimensional operators and allows us to manipulate instead matrices. Although we state Theorems 3 and 4 for general Wiener chaoses, it is sufficient to establish them on $\mathcal{W}_m^{(0)}$ instead of $\mathcal{W}_m$. Working in this finite setting, we show that, for all $q \geq 1$, there exists $\delta_q, C_q > 0$ such that

$$d_{\mathrm{FM}}\big(F, \mathcal{N}(0, 1)\big) \leq \delta_q \quad \Rightarrow \quad \mathbf{E}\big[\Gamma[F, F]^{-q}\big] \leq C_q, \quad F \in \mathcal{W}_m^{(0)}. \tag{14}$$

Let us show that by density of $\mathcal{W}_m^{(0)}$ in $\mathcal{W}_m$, (14) is actually sufficient to conclude on $\mathcal{W}_m$. Indeed, by continuity of $F \mapsto d_{\mathrm{FM}}(F, \mathcal{N}(0, 1))$ with respect to the $L^2(\mathbf{P})$-topology, we find that the condition on the left-hand side of (14) is $L^2(\mathbf{P})$-closed. Moreover, for $F_n \to F$ in

$L^2(\mathbf{P})$, on $\mathcal{W}_{\leq m}$, $\Gamma[F_n, F_n] \to \Gamma[F, F]$ in L^2, and thus, up to extraction, convergence almost sure. By Fatou's lemma we have

$$\mathbf{E}\big[\Gamma[F, F]^{-q}\big] \leq \liminf_{n\to\infty} \mathbf{E}\big[\Gamma[F_n, F_n]^{-q}\big] \leq C_q.$$

We often work with polynomial random variables living finite sums of chaoses. Thus, we define

$$\mathcal{W}_{\leq m}^{(0)} := \bigoplus_{k=0}^{m} \mathcal{W}_k^{(0)}.$$

3.2. *Malliavin calculus.* We introduce the operators coming from Malliavin's calculus used in this article. We refer to the textbooks [36, 41] for a broader introduction. Since we work with polynomial mappings evaluated in finite-dimensional Gaussian vectors, we skip any technical considerations regarding domain and integrability that frequently appear in Malliavin calculus.

3.2.1. *The square field operator.* We fix a Gaussian vector $\vec{N} = (N_1, \dots, N_K)$. Given two multivariate polynomial mappings F and $\tilde{F} \in \mathbb{R}[N_1, \dots, N_K]$, the *square field operator* of F and $\tilde{F}$ is

$$\Gamma[F, \tilde{F}] := \sum_{i=1}^{K} \frac{\partial F}{\partial N_i}(\vec{N}) \frac{\partial \tilde{F}}{\partial N_i}(\vec{N}). \tag{15}$$

If $\vec{F} = (F_1, \dots, F_d)$ is a random vector whose coordinates are as above, the Malliavin matrix $\Gamma(\vec{F})$ is the $d \times d$ positive symmetric random matrix defined by $\Gamma(\vec{F})_{i,j} := \Gamma[F_i, F_j]$.

When considering F and $\tilde{F} \in \mathbb{R}[N_1, \dots, N_K, G_1, \dots, G_{K'}]$, where $\vec{N}$ and $\vec{G}$ are independent standard Gaussian vectors, we set

$$\Gamma_N[F, F] := \sum_{i=1}^{K} \frac{\partial F}{\partial N_i}(\vec{N}, \vec{G}) \frac{\partial \tilde{F}}{\partial N_i}(\vec{N}, \vec{G}), \tag{16}$$

$$\Gamma_G[F, F] := \sum_{i=1}^{K'} \frac{\partial F}{\partial G_i}(\vec{N}, \vec{G}) \frac{\partial \tilde{F}}{\partial G_i}(\vec{N}, \vec{G}). \tag{17}$$

In this way, $\Gamma[F, F] = \Gamma_N[F, F] + \Gamma_G[F, F]$. Similarly, when $\vec{F} = (F_1, \dots, F_d)$ is a random vector, the conditional Malliavin matrices $\Gamma_N(\vec{F})$ and $\Gamma_G(\vec{F})$ are defined by $\Gamma_N(\vec{F})_{i,j} := \Gamma_N[F_i, F_j]$ and $\Gamma_G(\vec{F})_{i,j} := \Gamma_G[F_i, F_j]$. We recall that $\Gamma(F, \tilde{F}) = \langle \mathsf{D}F, \mathsf{D}\tilde{F}\rangle$. By [41], Proposition 1.2.2, D is continuous on $\mathcal{W}_{\leq m}$, and thus the operators Γ thus defined can be extended by density to $\mathcal{W}_{\leq m}$ for all $m \in \mathbb{N}$.

3.2.2. *Malliavin's lemma.* As a consequence of the seminal work of Malliavin [30] concerning the proof of the Hörmander criterion, a random vector $\vec{F}$ of the Wiener space, which is sufficiently smooth in some sense and such that $\det(\Gamma(\vec{F}))$ has negative moments at any order, has a smooth density. Moreover, it is a quantitative statement enabling to bound uniform norms of the derivatives of the densities with respect to negative moments of the determinant of the Malliavin matrix. In the framework of random vectors whose components are in a finite sum of Wiener chaoses, this result takes the following simpler form.

LEMMA 23. *Let m and $d \in \mathbb{N}^*$, and $q \in \mathbb{N}$. Then there exist $q' \in \mathbb{N}$ and $C > 0$, both depending only on (m, d, q) such that*

$$\|f\|_{W^{q,1}(\mathbb{R}^d)} \leq C\ \mathbf{E}\big[\det \Gamma(\vec{F})^{-q'}\big]$$

for any random vector $\vec{F} \in \mathcal{W}^d_{\leq m}$ *with its Euclidean norm* $\|\vec{F}\|$ *normalized such that* $\mathbf{E}[\|\vec{F}\|^2] = d$ *and whose corresponding density is denoted by* f.

PROOF. This lemma is nowadays rather standard in Malliavin calculus and relies on successive integrations by parts. References [17], Theorem 2.2, or [16], Theorem 3.2, provide similar statements. The exact statement of the theorem comes from [41], Proposition 2.1.4, with the choice $G = 1$ and $u_j = \mathsf{D}F_j$; see the paragraph after the proof, in particular [41], equation (2.32), and the subsequent equation. □

3.2.3. *Normal convergence and carré du champ on Wiener chaoses.* We repeatedly call upon the following emblematic result from the literature of limit theorems for Wiener chaoses: a sequence of chaotic random variables is asymptotically normal if and only if its carré du champ converges to a constant. We state the most general version for vectors.

THEOREM 24 ([35, 38, 42]). *Let* $d \in \mathbb{N}^*$, *and* $m_1, \dots, m_d \in \mathbb{N}^*$. *Then*

$$\vec{F}_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, I_d) \iff \Gamma(\vec{F}_n) \xrightarrow[n\to\infty]{L^2(\mathbf{P})} \operatorname{diag}(m_1, \dots, m_d), \quad (\vec{F}_n) \subset \prod_i \mathcal{W}_{m_i}.$$

Moreover, with $C > 0$ *a constant depending only on* $m_1, \dots, m_d$,

$$d_{\mathrm{FM}}\big(\vec{F}, \mathcal{N}(0, I_d)\big) \leq C \big\|\Gamma(\vec{F}) - \operatorname{diag}(m_1, m_2, \dots, m_d)\big\|_{L^2}, \quad \vec{F} \in \prod_i \mathcal{W}_{m_i}.$$

4. The second Wiener chaos. In this section we study the case of the second Wiener chaos $\mathcal{W}_2$. Since to every element $F \in \mathcal{W}_2$ corresponds a symmetric quadratic form, reduction theory makes the analysis much easier. We present here some of these tools, and we use them to prove Theorem 3 in this simpler context. The proof in the general case relies in a crucial way on the result for $\mathcal{W}_2$.

4.1. *Quadratic forms.*

4.1.1. *Diagonalization.* Every element $F \in \mathcal{W}_2^{(0)}$ is of the form

$$F = \sum_{i,j=1}^{K} a_{i,j} X_{i,j}, \quad X_{i,j} := \begin{cases} N_i N_j, & \text{if } i \neq j, \\ N_i^2 - 1, & \text{if } i = j, \end{cases}$$

for some $K \in \mathbb{N}^*$ and where $A = (a_{i,j})$ is a symmetric matrix of size $K \times K$. Thus, defining

$$q(x, x) := \sum_{i,j=1}^{K} a_{i,j} x_i x_j, \quad x \in \mathbb{R}^K,$$

a diagonalization procedure yields

$$F = q(\vec{N}, \vec{N}) - \mathbf{E}\big[q(\vec{N}, \vec{N})\big] = \sum_{i=1}^{K} \lambda_i \big(\tilde{N}_i^2 - 1\big),$$

where (λ_i) is the spectrum of A, and $(\tilde{N}_i)$ is a new sequence of independent standard Gaussian variables, obtained by an orthogonal transformation of (N_k).

4.1.2. *Spectral considerations.* The following spectral quantities play a crucial role in our proofs. Given a symmetric matrix A with spectrum $\{\lambda_i\}$, we define its *spectral remainders*:

$$\mathcal{R}_q(A) := \sum_{i_1 \neq i_2 \neq \cdots \neq i_q} \lambda_{i_1}^2 \cdots \lambda_{i_q}^2, \quad q \in \mathbb{N}^*. \tag{18}$$

In all the paper, the above notation $i_1 \neq i_2 \neq \cdots \neq i_q$ indicates summation over all pairwise distinct indices.

We conveniently generalize the definition of $\mathcal{R}_q$ to nonsquare matrices. In this case we replace the spectrum by the singular values of A. Namely,

$$\mathcal{R}_q(A) := \sum_{i_1 \neq i_2 \neq \cdots \neq i_q} \mu_{i_1} \cdots \mu_{i_q}, \quad q \in \mathbb{N}^*, \tag{19}$$

where $\{\mu_i\}$ is the spectrum of ${}^tAA$. We highlight that $\mathcal{R}_q(A) = \mathcal{R}_q({}^tA)$. The generalized Cauchy–Binet formula [18] expresses $\mathcal{R}_q(A)$ in terms of extracted determinants,

$$\mathcal{R}_q(A) = \sum_{|I|=|J|=q} \det(A_{I,J})^2, \tag{20}$$

where $A_{I,J}$ is the extracted matrix $A_{I,J} := (a_{i,j})_{i\in I, j\in J}$.

If A is symmetric with spectrum ordered by decreasing absolute values $|\lambda_1| \geq |\lambda_2| \geq \cdots$, we define the *squared distance to matrices of rank less q*

$$r_q(A) := \inf\{\|A - B\|^2 : \mathrm{rk}(B) \leq q - 1\} = \sum_{i \geq q} \lambda_i^2, \quad q \in \mathbb{N}^*$$

with $\|\cdot\|$ designating the Euclidean norm on matrices.

PROOF OF THE EQUALITY. This is a consequence of the Eckart–Young–Mirsky Theorem for the Frobenius norm [21], 7.4.15. This theorem states that the orthogonal projection of A on the matrices of rank less or equal than q is

$$\mathrm{proj}_{\leq q}(A) = P D_q {}^t P,$$

where $D_q := \mathrm{diag}(\lambda_1, \dots, \lambda_q, 0 \dots, 0)$, with (λ_i) the spectrum of A, and P is the orthogonal matrix that diagonalizes A. $\square$

Since $\mathcal{R}_q(A) = 0$ if and only if A has rank less than q, in some sense $\mathcal{R}_q(A)$ also measures the distance of A to matrices with rank $q - 1$ or less. Actually, the two quantities $\mathcal{R}_q(A)$ and $r_q(A)$ are comparable.

LEMMA 25. *Let $q \in \mathbb{N}$ and A be a symmetric matrix,*

$$\mathcal{R}_{q-1}(A) r_q(A) \leq \mathcal{R}_q(A) \leq q \mathcal{R}_{q-1}(A) r_q(A); \tag{21}$$

$$\prod_{i=1}^{q} r_i(A) \leq \mathcal{R}_q(A) \leq q! \prod_{i=1}^{q} r_i(A); \tag{22}$$

$$r_q(A)^q \leq \mathcal{R}_q(A) \leq q! r_1(A)^{q-1} r_q(A). \tag{23}$$

PROOF. We only prove (21), since (22) proceeds from an immediate induction, and (23) follows from the monotony of $(r_q(A))_q$. For the second inequality in (21), we write

$$\begin{aligned}\mathcal{R}_q(A) &= q! \sum_{i_1<i_2<\cdots<i_q} \lambda_{i_1}^2 \cdots \lambda_{i_q}^2 \\ &= q! \sum_{i_1<i_2<\cdots<i_{q-1}} \lambda_{i_1}^2 \cdots \lambda_{i_{q-1}}^2 \underbrace{\sum_{i_q>i_{q-1}} \lambda_{i_q}^2}_{\leq r_q(A)} \\ &\leq q!\Big(\sum_{i_1<i_2<\cdots<i_{q-1}} \lambda_{i_1}^2 \cdots \lambda_{i_{q-1}}^2\Big) r_q(A) = q\mathcal{R}_{q-1}(A) r_q(A).\end{aligned}$$

For the first inequality, we have

$$\mathcal{R}_q(A) = \sum_{i_1\neq\cdots\neq i_{q-1}} \lambda_{i_1}^2 \cdots \lambda_{i_{q-1}}^2 \underbrace{\sum_{i_q\notin\{i_1,\dots,i_{q-1}\}} \lambda_{i_q}^2}_{\geq r_q(A)} \geq \mathcal{R}_{q-1}(A) r_q(A).$$

This completes the proof. □

Define the *spectral radius* of A, $\rho(A) := \max_{\lambda\in\mathrm{spec}(A)}|\lambda|$. From the above estimates, we deduce the following useful estimate.

LEMMA 26. *Let A be a $n \times n$ symmetric matrix such that* $\mathrm{Tr}(A^2) = 1$ *and* $\rho(A) \leq \frac{1}{q}$, *then*

$$\mathcal{R}_q(A) \geq \prod_{k=1}^{q-1} \big(1 - k\rho(A)\big).$$

PROOF. Writing as before $\lambda_1, \dots, \lambda_n$ for the eigenvalues of A, we have $\sum_{i=1}^n \lambda_i^2 = 1$ and $\sup_i |\lambda_i| = \rho(A)$, so $r_i(A) \geq 1 - (i-1)\rho(A)$, and the result follows from (22). □

4.2. *Proof of the main theorem on the second Wiener chaos.* With the spectral tools introduced above, we now establish Theorem 3 for elements of $\mathcal{W}_2$. Theorem 3 follows from the following more general estimate.

PROPOSITION 27. *Let $q \in \mathbb{N}$. There exists $C > 0$ such that*

$$\mathbf{E}\big[\Gamma[F,F]^{-q}\big] \leq C\mathcal{R}_{2q+1}(A)^{-\frac{1}{2}}, \quad F \in \mathcal{W}_2^{(0)}.$$

PROOF. We assume that $q \neq 0$; otherwise, the claim is trivial. Let $F \in \mathcal{W}_2^{(0)}$. Consider the matrix A associated to F through the quadratic form. A is of size $K \times K$ for some $K \in \mathbb{N}$ and has eigenvalues $(\lambda_k)_{1\leq k\leq K}$. Diagonalizing A, we can assume that

$$F = \sum_{k=1}^K \lambda_k \big(N_k^2 - 1\big).$$

Fix $t \in \mathbb{R}$. It follows that

$$\Gamma[F,F] = \sum_{k=1}^K 4\lambda_k^2 N_k^2;$$

$$\mathbf{E}\big[\mathrm{e}^{-\frac{t^2}{2}\Gamma[F,F]}\big] = \prod_{i=1}^K \frac{1}{(1+4t^2\lambda_i^2)^{1/2}}.$$

Expanding the product gives the trivial bound $\prod_{i=1}^{K}(1+t^2\lambda_i^2) \geq t^{2q}\mathcal{R}_q(A)$. Thus, we get

$$\mathbf{E}\big[\mathrm{e}^{-\frac{t^2}{2}\Gamma[F,F]}\big] \leq \frac{1}{t^q \cdot \mathcal{R}_q(A)^{1/2}}.$$

Using that

$$x^{-q} = c_q \int_0^\infty t^{q-1}\mathrm{e}^{-tx}\,\mathrm{d}t, \quad x > 0,\ c_q := \frac{1}{(q-1)!}, \tag{24}$$

we find that,

$$\mathbf{E}\big[\Gamma[F,F]^{-q}\big] = c_q \int_{\mathbb{R}} |t^{2q-1}|\,\mathbf{E}\big[\mathrm{e}^{-\frac{t^2}{2}\Gamma[F,F]}\big]dt \leq \frac{c_q}{\mathcal{R}_{2q+1}(A)^{1/2}}.$$

The result follows. □

We now complete the proof in the case of the second Wiener chaos.

PROPOSITION 28. *Theorem 3 holds for $m = 2$. In particular, a sequence $(F_n) \subset \mathcal{W}_2$ converging to a Gaussian distribution satisfies: for every $q \in \mathbb{N}$, there exist $N \in \mathbb{N}$ and $C > 0$ such that*

$$\mathbf{E}\big[\Gamma[F_n, F_n]^{-q}\big] \leq C, \quad n \geq N.$$

PROOF. We assume that $q \neq 0$; otherwise, the claim is trivial. As explained in Section 3.1.4, it is sufficient to prove our claims on $\mathcal{W}_2^{(0)}$. By density, without loss of generality we assume that $(F_n) \subset \mathcal{W}_2^{(0)}$. We denote by A_n the associated matrix of size $k_n \times k_n$ and $(\lambda_{i,n})$ its spectrum. By Proposition 27 it is sufficient to bound from below the quantities $\mathcal{R}_q(A_n)$. By assumption we have

$$\begin{cases} \mathbf{E}[F_n^2] \xrightarrow[n\to\infty]{} 1, \\ \mathbf{E}[F_n^4] \xrightarrow[n\to\infty]{} 3; \end{cases} \iff \begin{cases} \displaystyle\sum_{i=1}^{k_n} \lambda_{i,n}^2 \xrightarrow[n\to\infty]{} \frac{1}{2}, \\ \displaystyle\sum_{i=1}^{k_n} \lambda_{i,n}^4 \xrightarrow[n\to\infty]{} 0. \end{cases}$$

This implies that $\rho(A_n) \to 0$. Since, by Lemma 26, $\mathcal{R}_q(A_n) \geq \prod_{k=1}^{q-1}(1-k\rho(A_n))$, we deduce that $\mathcal{R}_q(A_n)$ is bounded by below for n large, and we conclude. □

5. Sharp operator. In this section we establish estimates regarding Malliavin derivatives, when we specifically choose to take derivatives in the directions of a Gaussian space. In this case the Malliavin derivative is an element of an enlarged Wiener space. Thus, we can use Gaussian analysis to conclude. Through these estimates, we obtain that negative moments of $\Gamma[F, F]$ are estimated by the spectral remainders of the Malliavin Hessian of D^2F. The estimates of this section are akin to our results from [17] obtained in a more general setting. For the sake of completeness, we present self-contained arguments tailor-made to the case of normal convergence on Wiener chaoses. For simplicity, we state our results in finite sums of Wiener chaos, they still hold for more general random variables by density arguments, similar to that of Section 3.1.4.

5.1. *Iterated sharp operators.* The *sharp operator*, introduced by Bouleau [9] with a slightly different definition, is a convenient way to interpret the Malliavin derivative. For a standard Gaussian vector $\vec{N} = (N_1, \dots, N_K)$ and a polynomial mapping $F \in \mathbb{R}[N_1, N_2, \dots, N_K]$, the *Bouleau derivative* of F is

$$\sharp[F] := \sum_{i=1}^{K} \frac{\partial F}{\partial N_i}(\vec{N}) G_i, \tag{25}$$

where $\vec{G} = (G_1, \dots, G_K)$ is a Gaussian vector independent of $\vec{N}$. This operator intimately relates to the square-field operator through the Laplace–Fourier identity

$$\mathbf{E}\big[e^{it\sharp[F]}\big] = \mathbf{E}\bigg[\exp\bigg(-\frac{t^2}{2}\Gamma[F, F]\bigg)\bigg].$$

Our work exploits other connections between F, $\Gamma[F, F]$, and $\sharp[F]$, and they will become apparent to the reader in the rest of this work. Intuitively, the random variable $\sharp[F]$ is simpler than F, in view of the independence between the terms G_i and $\frac{\partial F}{\partial N_i}(\vec{N})$.

We generalize the definition to cover iterated Malliavin derivatives. We fix $(G_{i,j})$ a sequence of independent standard Gaussian variables, independent of $\vec{N}$. For a polynomial function $F \in \mathbb{R}[N_1, \dots, N_K]$, we let

$$\sharp^k[F] := \sum_{1 \le i_1, \dots, i_k \le K} \frac{\partial^k F}{\partial N_{i_1} \cdots \partial N_{i_k}}(\vec{N}) G_{1,i_1} \cdots G_{k,i_k}, \quad k = 1, 2, \dots.$$

When $k = 1$, the definition of $\sharp^k$ is consistent with that of $\sharp$. When $k = 0$, the above formula is understood as $\sharp^0[F] = F$. By density the operators $\sharp^k$ extends to $\mathcal{W}_m$ (to see this, observe that $\sharp^k$ is simply the iterated Malliavin derivative when we choose $\mathfrak{H}$ to be a Gaussian space and conclude by [36], Proposition 2.3.4). We regard $\sharp^k[F]$ as an element of an enlarged Wiener space, in the sense of (12), generated by the variables (N_k) and $(G_{i,j})$.

These operators satisfy the following elementary properties:

1. $F \in \mathcal{W}_m \Rightarrow \sharp^k[F] \in \mathcal{W}_m$, and $F \in \mathcal{W}_{\le m} \Rightarrow \sharp^k[F] \in \mathcal{W}_{\le m}$.
2. $F \in \mathcal{W}_m \Rightarrow \mathbf{Var}[\sharp^k[F]] = m(m-1)\cdots(m-k+1))\,\mathbf{Var}[F]$.

The first point follows immediately, since, for $F \in \mathcal{W}_m$, the partial derivatives with respect to the N_i's are in $\mathcal{W}_{m-1}$ and in view of the independence of $\vec{N}$ and $\vec{G}$. For the second point, we use a consequence of the integration by part formula, which gives for $F \in \mathcal{W}_m$; see, for instance, [41], Proposition 1.2.2, that

$$\mathbf{E}\big[F^2\big] = \frac{1}{m}\sum_{i=1}^{K} \mathbf{E}\bigg[\bigg(\frac{\partial F}{\partial N_i}\bigg)^2\bigg] = \frac{1}{m}\mathbf{E}\big[\sharp[F]^2\big].$$

Iterating this formula gives the second point.

5.2. *Negative moments estimates.* Estimates on the negative moments of $\Gamma[\sharp^k[F], \sharp^k[F]]$ yield estimates on those of $\Gamma[F, F]$.

PROPOSITION 29. *Let m and $k \in \mathbb{N}^*$ with $k \le m$. For all $q \in \mathbb{N}$, there exists $q' \in \mathbb{N}$ and $C > 0$ such that, for every $F \in \mathcal{W}_{\le m}$ with $\mathbf{E}[F^2] = 1$,*

$$\mathbf{E}\big[\Gamma[F, F]^{-q}\big] \le C\,\mathbf{E}\big[\Gamma\big[\sharp^k[F], \sharp^k[F]\big]^{-q'}\big].$$

PROOF. Let $F \in \mathcal{W}_{\leq m}$. Write $F_k = \sharp^k[F]$ for $k = 0, \dots, m$. For $k \geq 1$, the following induction relation holds:

$$F_k = \sum_{i=1}^{K} \frac{\partial F_{k-1}}{\partial N_i} G_{i,k}.$$

This shows that F_k has the same law as $V_k^{\frac{1}{2}} N$, where

$$V_k = \sum_{i=1}^{K} \left(\frac{\partial F_{k-1}}{\partial N_i} \right)^2 = \Gamma_N[F_{k-1}, F_{k-1}]$$

and $N \sim \mathcal{N}(0, 1)$ is independent of V_k. In particular, we have the Fourier–Laplace identity

$$\mathbf{E}\big[\mathrm{e}^{\mathrm{i}t F_k}\big] = \mathbf{E}\big[\mathrm{e}^{-\frac{t^2}{2}\Gamma_N[F_{k-1}, F_{k-1}]}\big]. \tag{26}$$

Fix m, q, and k as in the theorem. In Malliavin's result (Lemma 23), take the $q' \in \mathbb{N}$ associated with $2q+1$. If $\mathbf{E}[\Gamma[F_k, F_k]^{-q'}] = \infty$, then the statement is empty, and the proof is complete. Thus, we assume that $\mathbf{E}[\Gamma[F_k, F_k]^{-q'}] < \infty$. By Lemma 23 the density f_k of F_k belongs to $W^{2q+1,1}(\mathbb{R})$, and

$$\| f_k \|_{W^{2q+1,1}(\mathbb{R})} \leq C\, \mathbf{E}\big[\Gamma[F_k, F_k]^{-q'}\big].$$

(4) gives the bound on the Fourier transform,

$$\| f_k \|_{N^j} | \leq \| f_k \|_{W^{2q+1,1}(\mathbb{R})}, \quad j \leq 2q+1.$$

Therefore, up to changing the constant C,

$$\big| t^{2q-1}\, \mathbf{E}\big[\mathrm{e}^{\mathrm{i}t F_k}\big] \big| \leq \frac{C}{t^2+1} \mathbf{E}\big[\Gamma[F_k, F_k]^{-q'}\big], \quad t \in \mathbb{R}.$$

Reporting in the Fourier–Laplace identity (26) yields

$$\big| t^{2q-1} \big| \, \mathbf{E}\left[\exp\left(-\frac{t^2}{2} \Gamma_N[F_{k-1}, F_{k-1}] \right) \right] \leq \frac{C}{t^2+1} \mathbf{E}\big[\Gamma[F_k, F_k]^{-q'}\big], \quad t \in \mathbb{R}.$$

Using (24), we find that

$$\mathbf{E}\big[\Gamma_N[F_{k-1}, F_{k-1}]^{-q}\big] = c_q \int_0^\infty |t|^{2q-1}\, \mathbf{E}\big[\mathrm{e}^{-\frac{t^2}{2}\Gamma_N[F_k, F_k]}\big]\, \mathrm{d}t \leq C\, \mathbf{E}\big[\Gamma[F_k, F_k]^{-q'}\big],$$

where Γ_N is defined in (16). Since

$$\Gamma[F_{k-1}, F_{k-1}] = \Gamma_N[F_{k-1}, F_{k-1}] + \Gamma_G[F_{k-1}, F_{k-1}] \geq \Gamma_N[F_{k-1}, F_{k-1}],$$

we deduce that

$$\mathbf{E}\big[\Gamma[F_{k-1}, F_{k-1}]^{-q}\big] \leq C\, \mathbf{E}\big[\Gamma[F_k, F_k]^{-q'}\big].$$

The statement follows by an immediate induction. □

Combining Proposition 29 for $k = 2$ with Proposition 27, we obtain the estimate for negative moments of $\Gamma[F, F]$ in terms of spectral remainders for $\nabla^2 F$, the Hessian matrix of F, that is,

$$\nabla^2 F := \left(\frac{\partial^2 F}{\partial N_i \partial N_j} \right)_{1 \leq i, j \leq K}, \quad F \in \mathbb{R}[N_1, \dots, N_K].$$

PROPOSITION 30. *Let $m \in \mathcal{N}_{>0}$ and $q \in \mathbb{N}$, there exist $q' \in \mathbb{N}$ and $C > 0$ such that, for every $F \in \mathbb{R}[N_1, \dots, N_K]$ of degree m with $\mathbf{Var}[F] = 1$,*

$$\mathbf{E}[\Gamma[F, F]^{-q}] \leq C\,\mathbf{E}[\mathcal{R}_{q'}(\nabla^2 F)^{-\frac{1}{2}}].$$

PROOF. Fix $m \in \mathbb{N}^*$ and $q \in \mathbb{N}$. Let us denote by $(A_{i,j})$ the Hessian matrix of F, and let $\widetilde{F} := \sharp^2[F]$. Then

$$\widetilde{F} = \sum_{i,j \geq 1} A_{i,j} G_{i,1} G_{j,2}.$$

By Proposition 29 there exist $q'' \in \mathbb{N}$ and $C > 0$ such that

$$\mathbf{E}[\Gamma[F, F]^{-q}] \leq C\,\mathbf{E}[\Gamma[\widetilde{F}, \widetilde{F}]^{-q''}].$$

Thus, it is sufficient to bound from above the right-hand side. Since A and $\vec{G}$ are independent, fixing a realization of the entries $A_{i,j}$, $\widetilde{F}$ is a function of the variables $G_{i,1}$ and $G_{i,2}$ and can be seen as an element of the second Wiener chaos, with associated Hessian matrix

$$\widetilde{A} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

The characteristic polynomial of $\widetilde{A}$ is $t \mapsto \chi_A(t)\chi_A(-t)$, where χ_A stands for the characteristic polynomial of A. Hence,

$$\operatorname{spec}(\widetilde{A}) = \{\lambda, -\lambda : \lambda \in \operatorname{spec}(A)\}.$$

This yields that $\mathcal{R}_p(A) \leq \mathcal{R}_p(\widetilde{A})$ ($p \in \mathbb{N}^*$). Applying Proposition 27 gives $q' := 2q'' + 1$ and $C > 0$ such that

$$\mathbf{E}_G[\Gamma_G[\widetilde{F}, \widetilde{F}]^{-q''}] \leq C\mathcal{R}_{q'}(\widetilde{A})^{-\frac{1}{2}} \leq C\mathcal{R}_{q'}(A)^{-\frac{1}{2}},$$

where $\mathbf{E}_G$ (resp., Γ_G) means that we only integrate (resp., derivate) with respect to the variables $G_{i,j}$. Using that $\Gamma[\widetilde{F}, \widetilde{F}] \geq \Gamma_G[\widetilde{F}, \widetilde{F}]$ and integrating with respect to N, we get

$$\mathbf{E}[\Gamma[\widetilde{F}, \widetilde{F}]^{-q''}] \leq C\,\mathbf{E}[\mathcal{R}_{q'}(A)^{-\frac{1}{2}}].$$

This concludes the proof. □

6. Proof of the main theorems.

6.1. *Setup*. In this section we prove Theorem 3. We proceed by induction on the degree m of the chaos. Let us define the property to be established.

$$(\mathcal{P}(m))\qquad \begin{aligned}&\text{For every sequence } (F_n)_{n\geq 1} \subset \mathcal{W}_m,\\ &\Big[F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1)\Big] \Rightarrow \Big[\limsup_{n\to+\infty} \mathbf{E}[\Gamma[F_n, F_n]^{-q}] < +\infty, q \in \mathbb{N}\Big].\end{aligned}$$

This property is equivalent to the following nonsequential version:

$$(\mathcal{P}(m))\qquad \begin{aligned}&\forall q \in \mathbb{N},\ \exists \delta = \delta_q > 0,\ \exists C = C_q > 0 :\\ &\forall F \in \mathcal{W}_m,\ [d_{\mathrm{FM}}(F, \mathcal{N}(0,1)) \leq \delta] \Rightarrow [\mathbf{E}[\Gamma[F, F]^{-q}] \leq C].\end{aligned}$$

Section 4 establishes $\mathcal{P}(2)$. Let us prove that, for every $m \geq 3$, $\mathcal{P}(m-1) \Rightarrow \mathcal{P}(m)$.

We often use that controls on negative moments in $\mathcal{P}(m)$ are expressible in terms of small ball estimates. More precisely, for every sequence of random variables (X_n), we recall the following elementary equivalence:

1. For every $q \geq 0$, there exists $N \in \mathbb{N}$ such that $\sup_{n\geq N} \mathbf{E}[|X_n|^{-q}] < +\infty$.
2. For every $q \geq 0$, there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$\forall \epsilon > 0,\ \sup_{n\geq N} \mathbf{P}[|X_n| \leq \epsilon] \leq C\epsilon^q.$$

6.2. *The discretization procedure.* Through a discretization procedure, we obtain that $\mathcal{P}(m)$ is equivalent to the following vectorial version. For $d \in \mathbb{N}^*$, we consider

$$(\mathcal{P}_d(m)) \qquad \begin{gathered}\text{For every sequence } (\vec{F}_n)_{n\geq 1} \subset \mathcal{W}^d_{\leq m},\\ \Big[\vec{F}_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, I_d)\Big] \Rightarrow \Big[\limsup_{n\to+\infty} \mathbf{E}\big[\det \Gamma(\vec{F}_n)^{-q}\big] < +\infty, \quad q \in \mathbb{N}\Big].\end{gathered}$$

As above, it is equivalent to the nonsequential version,

$$(\mathcal{P}_d(m)) \qquad \begin{gathered}\forall q \in \mathbb{N}, \exists \delta = \delta_q > 0, \exists C = C_q > 0 :\\ \forall \vec{F} \in \mathcal{W}^d_m, \big[d_{\mathrm{FM}}\big(\vec{F}, \mathcal{N}(0, I_d)\big) \leq \delta\big] \Rightarrow \big[\mathbf{E}\big[\det\big(\Gamma(\vec{F})\big)^{-q}\big] \leq C\big].\end{gathered}$$

In this section we prove the implication $\mathcal{P}(m) \Rightarrow [\forall d \in \mathbb{N}^*, \mathcal{P}_d(m)]$ via a more general statement.

PROPOSITION 31. *Let d and $m \in \mathbb{N}^*$. Consider a sequence $(\vec{F}_n) \subset \mathcal{W}^d_{\leq m}$ that is also L^2-bounded sequence. Then there is equivalence between the two following properties*:

(i) *For every sequence $(\vec{a}_n)$ in the sphere $\mathbb{S}^{d-1}$, and all $q > 0$*

$$\limsup_{n\to+\infty} \mathbf{E}\big[\Gamma[\vec{F}_n \cdot \vec{a}_n, \vec{F}_n \cdot \vec{a}_n]^{-q}\big] < +\infty.$$

(ii) *For all $q > 0$, $\limsup_{n\to+\infty} \mathbf{E}[\det \Gamma(\vec{F}_n)^{-q}] < +\infty$.*

COROLLARY 32. *For any $m \geq 2$, if $\mathcal{P}(m)$ holds, then $\mathcal{P}_d(m)$ also holds for every $d \in \mathbb{N}^*$.*

The proof of the proposition relies on a discretization procedure of the sphere. Such procedure is frequently used in Malliavin calculus, for instance [16], Lemma 4.7. We use the following discretization result for the $d-1$-dimensional Euclidean sphere $\mathbb{S}^{d-1}$.

LEMMA 33. *For all $d \in \mathbb{N} \setminus \{1\}$ and $N \in \mathbb{N}$, there exist $C_d > 0$ (not depending on N) and $\mathbb{S}^{d-1,N} \subset \mathbb{S}^{d-1}$ such that* $\mathrm{Card}(\mathbb{S}^{d-1,N}) \leq C_d N^d$ *and*

$$\forall a \in \mathbb{S}^{d-1}, \exists b \in \mathbb{S}^{d-1,N}, \text{ such that } \|a - b\| \leq \frac{C_d}{N}.$$

PROOF. We fix an positive integer N. Write $[\![-N, N]\!] := \{-N, -N+1, \dots, N-1, N\}$. For every $I = (i_1, i_2, \dots, i_d) \in [\![-N, N]\!]^d$, we set $b_I := \frac{I}{N}$. For every $a \in \mathbb{S}^{d-1}$, we may find $I \in [\![-N, N]\!]^d$ such that all the coordinates of $b_I - a$ are $\leq \frac{1}{N}$. Hence,

$$\big|\|b_I\| - 1\big| \leq \|b_I - a\| \leq \frac{\sqrt{d}}{N}.$$

Whenever $N \geq 2\sqrt{d}$, the above choice of b_I yields $\|b_I\| \geq \frac{1}{2}$. Thus, setting $a_I := \frac{b_I}{\|b_I\|}$, we get

$$\|a_I - a\| \leq 2\big\|b_I - a\|b_I\|\big\| \leq 2\|b_I - a\| + 2\big|1 - \|b_I\|\big| \leq \frac{4\sqrt{d}}{N}.$$

We define

$$\mathbb{S}^{d-1,N} := \left\{\frac{b_I}{\|b_I\|} = a_I : I \in [\![-N, N]\!]^d / \{0, \dots, 0\}\right\}.$$

We have proved that, provided that $N \geq 2\sqrt{d}$, for any $a \in \mathbb{S}^{d-1}$, we may find $a_I \in \mathbb{S}^{d-1,N}$ such that $\|a - a_I\| \leq \frac{4\sqrt{d}}{N}$. Besides $\mathrm{Card}(\mathbb{S}^{d-1,N}) \leq (2N+1)^d$. □

Let us prove the proposition.

PROOF OF PROPOSITION 31. In all the proof, we write indistinctly $\|\cdot\|$ for the Euclidean norm of a vector, or the Euclidean norm of a matrix, also known as its Hilbert–Schmidt norm. We first prove $(i) \Rightarrow (ii)$, which is the only implication used in the paper. Assuming (i), we want to obtain a bound

$$\mathbf{P}\big[\det \Gamma(\vec{F}_n) \le \epsilon\big] \le C_q \epsilon^q, \quad \varepsilon > 0,\ q \ge 0.$$

For any S symmetric positive matrix $d \times d$, we have

$$\inf_{a \in \mathbb{S}^{d-1}} {}^t\vec{a} S \vec{a} = \lambda_1(S) \le \det(S)^{\frac{1}{d}},$$

where $\lambda_1(S)$ is the smallest eigenvalue of S. Thus, it suffices to prove that for every $q \ge 0$ there exist $N \in \mathbb{N}$ and $C > 0$ such that, for $n \ge N$,

$$\mathbf{P}\Big[\inf_{\vec{a} \in \mathbb{S}^{d-1}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le \epsilon\Big] \le C\epsilon^q, \quad \varepsilon > 0.$$

Let N be an integer to be chosen later. By Lemma 33, for every $\vec{a} \in \mathbb{S}^{d-1}$ and $\vec{b} \in \mathbb{S}^{d-1,N}$, we have that $\|\vec{a} - \vec{b}\| \le \frac{C}{N}$. In this way

$$\big|\Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] - \Gamma[\vec{F}_n \cdot \vec{b}, \vec{F}_n \cdot \vec{b}]\big| = \big|{}^t(\vec{a} - \vec{b})\Gamma(\vec{F}_n)(\vec{a} + \vec{b})\big| \le \frac{2C}{N}\|\Gamma(\vec{F}_n)\|.$$

This gives

$$\inf_{\vec{a} \in \mathbb{S}^{d-1}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le \inf_{\vec{a} \in \mathbb{S}^{d-1,N}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] + \frac{2C}{N}\|\Gamma(\vec{F}_n)\|.$$

Consequently, we have

$$\Big\{\inf_{\vec{a} \in \mathbb{S}^{d-1}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le \epsilon\Big\} \subset \Big\{\inf_{a \in \mathbb{S}^{d-1,N}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le \epsilon + \frac{2CK}{N}\Big\} \cup \big\{\|\Gamma(\vec{F}_n)\| > K\big\}.$$

We choose $N = \lceil \frac{1}{\epsilon^2} \rceil$ and $K = \lceil \frac{1}{\epsilon} \rceil$ so that, for a different constant C,

$$\Big\{\inf_{\vec{a} \in \mathbb{S}^{d-1}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le \epsilon\Big\} \subset \Big\{\inf_{a \in \mathbb{S}^{d-1,N}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le C\epsilon\Big\} \cup \Big\{\|\Gamma(\vec{F}_n)\| > \frac{1}{\epsilon}\Big\}. \tag{27}$$

By assumption there exists $C > 0$ such that, for n large,

$$\sup_{\vec{a} \in \mathbb{S}^{d-1}} \mathbf{P}\big[\Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le \epsilon\big] \le C\epsilon^{q+2d}, \quad \epsilon > 0,$$

and we find, with another constant $C' > 0$,

$$\begin{aligned} \mathbf{P}\Big[\inf_{a \in \mathbb{S}^{d-1,N}} \Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le C\epsilon\Big] &\le \sum_{a \in \mathbb{S}^{d-1,N}} \mathbf{P}\big[\Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le C\epsilon\big] \\ &\le CN^d \sup_{a \in \mathbb{S}^{d-1}} \mathbf{P}\big[\Gamma[\vec{F}_n \cdot \vec{a}, \vec{F}_n \cdot \vec{a}] \le C\epsilon\big] \\ &\le C' \frac{1}{\epsilon^{2d}} \epsilon^{2d+q} = C'\epsilon^q. \end{aligned} \tag{28}$$

Since $(\vec{F}_n)$ is bounded in L^2 and $(\vec{F}_n) \subset \mathcal{W}^d_{\le m}$, $(\Gamma(\vec{F}_n))$ is also bounded in L^2 and then in L^q by equivalence of norms on Wiener chaoses Section 3.1.3. Markov inequality gives for some C,

$$\mathbf{P}\Big[\|\Gamma(\vec{F}_n)\| > \frac{1}{\epsilon}\Big] \le C\epsilon^q.$$

We conclude. For completeness, let us sketch the proof of the converse implication (ii) $\Rightarrow$ (i). In this case we start from the bound

$$\det(S) = \prod_{i=1}^{d} \lambda_i \leq \lambda_1 \|A\|^{d-1}.$$

Thus,

$$\inf_{\vec{a} \in \mathbb{S}^{d-1}} \vec{a} \cdot \Gamma(\vec{F}_n)\vec{a} = \lambda_1\big(\Gamma(\vec{F}_n)\big) \geq \det\big(\Gamma(\vec{F}_n)\big)\|\Gamma(\vec{F}_n)\|^{-(d-1)}.$$

Now, $(\Gamma(\vec{F}_n))$ is bounded in all the $L^p(\mathbf{P})$ $(p \neq \infty)$, in view on the assumptions on $(\vec{F}_n)$ and the equivalence of the norms on $\mathcal{W}_{\leq m}$ (Section 3.1.3). $\square$

6.3. *Normal approximation in smaller chaos.* In this section, starting from an element of a chaos $\mathcal{W}_m$ whose law is close to a normal law, we construct variables in $\mathcal{W}_{m-1}$ whose laws are also close to normal laws. This construction allows us to use the induction hypothesis in the proof of Theorem 3. We start with some notations.

DEFINITION 34. Let $F \in \mathbb{R}[N_1, \dots, N_K]$ be a polynomial. If $\vec{x} = (x_1, \dots, x_K)$ is a vector of $\mathbb{R}^K$, we denote by $\mathsf{D}_{\vec{x}} F$ the *directional derivative* following $\vec{x}$, namely,

$$\mathsf{D}_{\vec{x}} F := \sum_{k=1}^{K} x_k \frac{\partial F_n}{\partial N_k}.$$

If X is a matrix $K \times d$ with column vectors $(\vec{x}_1, \dots, \vec{x}_d)$, we write

$$\mathsf{D}_X F := (\mathsf{D}_{\vec{x}_1} F, \dots, \mathsf{D}_{\vec{x}_d} F).$$

If $F \in \mathcal{W}_m$, then $\mathsf{D}_{\vec{x}} F \in \mathcal{W}_{m-1}$ and $\mathsf{D}_X F \in \mathcal{W}_{m-1}^d$.

The following proposition states that if $F \in \mathcal{W}_m$ is close in law to the standard Gaussian $\mathcal{N}(0, 1)$ and if we choose X randomly with respect to the Gaussian measure, then $\mathsf{D}_X F \in \mathcal{W}_{m-1}^d$ is close in distribution to the Gaussian vector $\mathcal{N}(0, mI_d))$ with large probability. We write $\gamma_{K,d}$ for the standard Gaussian distribution on the matrices of size $K \times d$.

PROPOSITION 35. *Let $m \in \mathbb{N}^*$. Consider a sequence $(F_n) \subset \mathcal{W}_m^{(0)}$ such that, for all $n \in \mathbb{N}$, $F_n \in \mathbb{R}[N_1, \dots, N_{K_n}]$ for some $K_n \in \mathbb{N}^*$, and $F_n \to \mathcal{N}(0, 1)$ in law. Then for every $d \in \mathbb{N}^*$ and every $\epsilon > 0$,*

$$\gamma_{K_n,d}\big\{X \in M_{K_n,d}(\mathbb{R}) : d_{\mathrm{FM}}\big(\mathsf{D}_X F_n, \mathcal{N}(0, mI_d)\big) \geq \epsilon\big\} \xrightarrow[n\to\infty]{} 0.$$

REMARK 36. The result states that, for n sufficiently large, there exists a set of matrices X of large Gaussian measure and such that the laws of all the $\mathsf{D}_X F_n$'s are closed to a normal distribution with covariance independent of the X. It might seem contradictory, since by multiplying X by a scalar λ the distribution should change. Contrary to the finite dimensional case, the image of an infinite Gaussian measure by a nontrivial homothety is singular with respect to the initial measure. For instance, by the law of large number, the Borel set

$$A := \left\{(x_i) \in \mathbb{R}^{\mathbb{N}} : \frac{1}{n}\sum_{i=1}^{n} x_i^2 \xrightarrow[n\to\infty]{} 1\right\},$$

has full $\gamma^{\mathbb{N}}$-measure. However, the set

$$\lambda A := \{\lambda x : x \in A\}$$

has measure 0 as soon as $|\lambda| \neq 1$.

PROOF. Let (F_n) be as in the theorem. Let $(G_{i,j})_{i,j\geq 1}$ be a family of independent standard Gaussian variables, independent of $\vec{N}$. For every $n \in \mathbb{N}$, we define the random $K_n \times d$ matrix $\mathscr{G}_n$ and the random vector $\vec{V}_n$ by

$$\mathscr{G}_n := (G_{i,j})_{1\leq i\leq d, 1\leq j\leq K_n},$$
$$\vec{V}_n := \mathsf{D}_{\mathscr{G}_n} F_n = (V_{n,1}, \dots, V_{n,d}).$$

In the Wiener space generated by the variables $(G_{i,j})$ and (N_k), $\vec{V}_n \in \mathcal{W}_m^d$, and the coordinates read

$$V_{n,i} = \sum_{j=1}^{K_n} \frac{\partial F_n}{\partial N_j} G_{i,j}, \quad i = 1, \dots, d. \tag{29}$$

It follows that $\vec{V}_n$ has same law as $\Gamma[F_n, F_n]^{\frac{1}{2}} \vec{G}'$ where $\vec{G}'$ is a standard Gaussian vector independent of $\vec{N}$. Since by Theorem 24,

$$\Gamma[F_n, F_n] \xrightarrow[n\to+\infty]{L^2} m, \tag{30}$$

we deduce that

$$\vec{V}_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, mI_d).$$

Using Theorem 24, we obtain that

$$\Gamma(\vec{V}_n) \xrightarrow[n\to\infty]{L^2} m^2 I_d. \tag{31}$$

Consider the decomposition $\Gamma(\vec{V}_n) = \Gamma_G(\vec{V}_n) + \Gamma_N(\vec{V}_n)$, where $\Gamma_G(\vec{V}_n)$ (resp., $\Gamma_N(\vec{V}_n)$) is the Malliavin matrix of $\vec{V}_n$ with respect with the coordinates $G_{i,j}$ (resp., N_k), as defined in (16) and (17). From (29) we directly compute the matrix $\Gamma_G(\vec{V}_n)$,

$$\Gamma_G[V_{n,i}, V_{n,j}] = 0, \quad i \neq j;$$
$$\Gamma_G[V_{n,i}, V_{n,i}] = \sum_{j=1}^{K_n} \left(\frac{\partial V_{n,i}}{\partial G_{i,j}}\right)^2 = \sum_{j=1}^{K_n} \left(\frac{\partial F_n}{\partial N_j}\right)^2 = \Gamma[F_n, F_n], \quad i = j.$$

Thus, $\Gamma_G(\vec{V}_n) = \Gamma[F_n, F_n] I_d$. By (30) we obtain that $\Gamma_G(\vec{V}_n) \to mI_d$ in L^2. Combining with (31), we deduce that

$$\Gamma_N(\vec{V}_n) \xrightarrow[n\to\infty]{L^2} m(m-1)I_d. \tag{32}$$

Since $\mathsf{D}_X F_n$ depends only on the variables N_k's and not the $G_{i,j}$'s, we get

$$\Gamma(\mathsf{D}_X F_n) = \Gamma_N(\mathsf{D}_X F_n), \quad \text{for any deterministic } X \in M_{K_n,d}(\mathbb{R}).$$

Thus, we rewrite (32) as

$$\int_{M_{K_n,d}(\mathbb{R})} \|\Gamma(\mathsf{D}_X F_n) - m(m-1)I_d\|_{L^2}^2 \, d\gamma_{K_n,d}(X) \xrightarrow[n\to\infty]{} 0. \tag{33}$$

For $X \in M_{K_n,d}$, $\mathsf{D}_X F_n \in \mathcal{W}_{m-1}^d$ so that Theorem 24 gives a constant $C = C_m > 0$ such that

$$d_{\mathrm{FM}}(\mathsf{D}_X F_n, \mathcal{N}(0, mI_d))^2 \leq C \|\Gamma(\mathsf{D}_X F_n) - m(m-1)I_d\|_{L^2}^2. \tag{34}$$

Finally, combining (33) and (34) yields

$$\int_{M_{K_n,d}(\mathbb{R})} d_{\mathrm{FM}}(\mathsf{D}_X F_n, \mathcal{N}(0, mI_d))^2 \, d\gamma_{K_n,d}(X) \to 0,$$

and we conclude by Markov's inequality. $\square$

6.4. *A compressing argument.* If $F = F(N_1, \dots, N_K) \in \mathcal{W}_m^{(0)}$, we need to study the $K \times K$ matrix $A := \nabla^2 F$. To that extent, we fix a wisely-chosen $K \times q$ matrix X for a fixed q, and we study the $K \times q$ *compressed matrix* $B := AX$. We choose X in a way that B contains most of the information on A. At the same time, B is simpler to study since the dimension is reduced. This strategy appears in information theory under the name of "compressed sensing."

6.4.1. *Control of the spectral remainder of the compressed Hessian.* An elementary computation shows that the $q \times q$ matrix ${}^t BB = \Gamma(\mathsf{D}_X F)$, and we use the tools developed above in order to study this Malliavin matrix. We recall that we have define the spectral remainders of a rectangular matrix M in Section 4.1.2 in terms of the singular values and that we have

$$\mathcal{R}_q(M) = \mathcal{R}_q\big(({}^t MM)^{\frac{1}{2}}\big), \quad q \in \mathbb{N}^*.$$

LEMMA 37. *If $F = F(N_1, \dots, N_K)$ is polynomial and if $X \in M_{K,q}(\mathbb{R})$, then*

$$\mathcal{R}_q(\nabla^2 F X) = \det\big(\Gamma(\mathsf{D}_X F)\big).$$

PROOF. Let $X := (\vec{x}_1, \dots, \vec{x}_q) = (x_{i,j})_{i \le K, j \le q}$ for some $\vec{x}_j \in \mathbb{R}^K$ and $B := (\nabla^2 F)X$. Then

$$\frac{\partial(\mathsf{D}_{\vec{x}_j} F)}{\partial N_i} = \sum_{k=1}^{K} x_{k,j} \frac{\partial^2 F}{\partial N_i \partial N_k} = B_{i,j}, \quad i \le K,\ j \le q.$$

This shows that $\vec{\nabla}(\mathsf{D}_X F) = B$ and $\Gamma(\mathsf{D}_X F) = {}^t BB$.

Moreover, since ${}^t BB$ is a $q \times q$ matrix, $\mathcal{R}_q(B)$ is by definition the product of the spectral values of ${}^t BB$, so it's determinant. Thus,

$$\mathcal{R}_q(B) = \det({}^t BB) = \det\big(\Gamma(\mathsf{D}_X F)\big). \qquad \square$$

LEMMA 38. *Let m, p, and $q \in \mathbb{N}^*$, with $m \ge 3$. Assume the induction property $\mathcal{P}(m-1)$. Then there exists $C > 0$ such that for every $(F_n(N_1, \dots, N_{K_n})) \subset \mathcal{W}_m^{(0)}$ converging in law to the standard Gaussian distribution, then, for $n \in \mathbb{N}$, large enough, the set*

$$\mathcal{E}_n := \{X \in M_{K_n,q}(\mathbb{R}) : \mathbf{E}[\mathcal{R}_q(\nabla^2 F_n X)^{-p}] \le C\}$$

has $\gamma_{K_n,q}$-measure more than $\frac{2}{3}$.

PROOF. By Corollary 32, $\mathcal{P}_q(m-1)$ holds. In particular, for all $p > 0$, there exist $\varepsilon > 0$ and $C > 0$ such that, for any $\vec{V} \in \mathcal{W}_{m-1}^q$,

$$d_{\mathrm{FM}}\big(\vec{V}, \mathcal{N}(0, mI_q)\big) \le \varepsilon \Rightarrow \mathbf{E}\big[\det\big(\Gamma(\vec{V})\big)^{-p}\big] \le C. \tag{35}$$

Applying to $\vec{V} := \mathsf{D}_X F_n$, we find by Lemma 37

$$d_{\mathrm{FM}}\big(\mathsf{D}_X F_n, \mathcal{N}(0, mI_q)\big) \le \varepsilon \Rightarrow \mathbf{E}\big[\mathcal{R}_q(\nabla^2 F_n X)^{-p}\big] \le C.$$

By Proposition 35, for $n \in \mathbb{N}$, large enough, the set

$$\{X \in M_{K_n,q}(\mathbb{R}) : d_{\mathrm{FM}}\big(\mathsf{D}_X F_n, \mathcal{N}(0, mI_q)\big) \le \varepsilon\}$$

has $\gamma_{K_n,q}$-measure more than $2/3$. This concludes the proof. $\square$

6.4.2. *Relating the spectral remainder of the compressed Hessian and the Hessian.* In this step we derive estimates on $\mathcal{R}_q(A)$ from estimates on $\mathcal{R}_q(AX)$ for a *generic* matrix X.

LEMMA 39. *For every $p, q \in \mathbb{N}^*$, there exists $C > 0$ such that, for every $d \times d$ symmetric matrix M,*

$$\mathbf{E}\big[\mathcal{R}_q(M\mathcal{X})^p\big] \le C\mathcal{R}_q(M)^p,$$

where $\mathcal{X}$ is a $d \times q$ matrix whose entries are independent standard Gaussian variables.

PROOF. Let us write $M = {}^tP\Delta P$ with P orthogonal and Δ diagonal, with diagonal values $\lambda_1, \dots, \lambda_d$. We have $\mathcal{R}_q(M) = \mathcal{R}_q(\Delta)$. Also, since $P\mathcal{X}$ and $\mathcal{X}$ have same law, we find that

$${}^t(M\mathcal{X})M\mathcal{X} = {}^t(P\mathcal{X})\Delta^2 P\mathcal{X} \overset{\text{Law}}{=} {}^t\mathcal{X}\Delta^2\mathcal{X}.$$

In particular, be definition of the spectral remainders for rectangular matrix, $\mathbf{E}[\mathcal{R}_q(M\mathcal{X})^p] = \mathbf{E}[\mathcal{R}_q(\Delta\mathcal{X})^p]$. Thus, we assume that $M = \Delta$. The entries of $\Delta\mathcal{X}$ are given by $(\Delta\mathcal{X})_{i,j} = \lambda_i\mathcal{X}_{i,j}$. Thus, for any subsets I, J of cardinality q, the extracted determinant on $I \times J$ is $\prod_{i\in I}\lambda_i \det(\mathcal{X}_{I,J})$. By the Cauchy–Binet formula (20),

$$\mathcal{R}_q(\Delta\mathcal{X}) = \sum_{|I|=q}\prod_{i\in I}\lambda_i^2 S_I, \quad \text{where } S_I := \sum_{|J|=q}\det(\mathcal{X}_{I,J})^2.$$

The variables S_I have same law and, in particular, same expectation c. This gives

$$\mathbf{E}\big[\mathcal{R}_q(\Delta\mathcal{X})\big] = c\sum_{|I|=q}\prod_{i\in I}\lambda_i^2 = c\mathcal{R}_q(\Delta).$$

The claim follows for $p = 1$. For $p \neq 1$, we use the equivalence of norms (13); since $\mathcal{R}_q(\Delta\mathcal{X})$ is a positive polynomial of degree q in Gaussian variables, there exists $C = C_{p,q}$ such that

$$\mathbf{E}\big[\mathcal{R}_q(\Delta\mathcal{X})^p\big] \le C\,\mathbf{E}\big[\mathcal{R}_q(\Delta\mathcal{X})\big]^p = C\mathcal{R}_q(\Delta)^p,$$

and the result follows. □

LEMMA 40. *Let $p, q \in \mathbb{N}^*$. There exists $C > 0$ such that, for every $K \in \mathbb{N}^*$ and every random symmetric matrix A in $M_{K,K}(\mathbb{R})$, the set*

$$\mathcal{E} := \left\{X \in M_{K,q}(\mathbb{R}) : \mathbf{E}\left[\frac{\mathcal{R}_q(AX)^p}{\mathcal{R}_q(A)^p}\right] \le C\right\}$$

has $\gamma_{K,q}$-measure more than $\frac{2}{3}$.

PROOF. By Lemma 39 there exists $C = C_{p,q}$ such that

$$\mathcal{R}_q(A)^p \ge C\int_{M_{K,q}(\mathbb{R})}\mathcal{R}_q(AX)^p\,\mathrm{d}\gamma_{K,q}(X).$$

Thus,

$$\int_{M_{K,q}(\mathbb{R})}\mathbf{E}\left[\frac{\mathcal{R}_q(AX)^p}{\mathcal{R}_q(A)^p}\right]\mathrm{d}\gamma_{K,q}(X) \le \mathbf{E}\left[\frac{1}{\mathcal{R}_q(A)^p} \times C\mathcal{R}_q(A)^p\right] = C.$$

As a result, we obtain, by Markov inequality, that

$$\gamma_{K,q}\left\{X \in M_{K,q}(\mathbb{R}) : \mathbf{E}\left[\frac{\mathcal{R}_q(AX)^p}{\mathcal{R}_q(A)^p}\right] \geq 3C\right\}$$

$$(36) \qquad \leq \frac{1}{3C}\int_{M_{K,q}(\mathbb{R})} \mathbf{E}\left[\frac{\mathcal{R}_q(AX)^p}{\mathcal{R}_q(A)^p}\right] \mathrm{d}\gamma_{K,q}(X)$$

$$\leq \frac{1}{3}.$$

The proof is complete. □

6.5. *Proof of the induction step.* PROOF OF THEOREM 3. We establish the induction step $\mathcal{P}(m-1) \Rightarrow \mathcal{P}(m)$. Let $m \in \mathbb{N}$, $m \geq 3$, and assume $\mathcal{P}(m-1)$. We fix a sequence $(F_n) \subset \mathcal{W}_m^{(0)}$ converging in law to $\mathcal{N}(0,1)$. As before, we assume that $F_n \in \mathbb{R}[N_1, \dots, N_{K_n}]$, and we set $A_n := \nabla^2 F_n$, which is a random matrix of size $K_n \times K_n$. Fix p and $q \in \mathbb{N}^*$, and fix $n \in \mathbb{N}$ large enough for Lemma 38 to apply to F_n. Thus, there exists $\mathcal{E}_1 \subset M_{K_n,q}(\mathbb{R})$ of $\gamma_{K_n,q}$-measure more than $\frac{2}{3}$ such that

$$(37) \qquad \mathbf{E}\left[\mathcal{R}_q(A_n X)^{-p}\right] \leq C_1, \quad X \in \mathcal{E}_1,$$

where $C_1 > 0$ depends only on m, p, and q. By Lemma 40 there exists $\mathcal{E}_2 \subset M_{K_n,q}(\mathbb{R})$ of $\gamma_{K_n,q}$-measure more than $\frac{2}{3}$ such that

$$(38) \qquad \mathbf{E}\left[\frac{\mathcal{R}_q(A_n X)^p}{\mathcal{R}_q(A)^p}\right] \leq C_2, \quad X \in \mathcal{E}_2,$$

where C_2 depends only on p and q. Since $\gamma_{K_n,q}(\mathcal{E}_1 \cap \mathcal{E}_2) \geq \frac{1}{3}$, the sets $\mathcal{E}_1$ and $\mathcal{E}_2$ have a nonempty intersection. In particular, there exists $X \in M_{K_n,q}(\mathbb{R})$ such that estimates (37) and (38) hold simultaneously. Then by Cauchy–Schwarz inequality

$$\mathbf{E}\left[\mathcal{R}_q(A_n)^{-\frac{p}{2}}\right]^2 \leq \mathbf{E}\left[\mathcal{R}_q(A_n X)^{-p}\right]\mathbf{E}\left[\frac{\mathcal{R}_q(A_n X)^p}{\mathcal{R}_q(A_n)^p}\right] \leq C_1 C_2.$$

Since in the previous argument, p and q are arbitrary positive integers, we have shown

$$\limsup_{n\to+\infty} \mathbf{E}\left[\mathcal{R}_q(A_n)^{-p}\right] < +\infty, \quad p, q \in \mathbb{N}^*.$$

Specifying the above estimate to $p = \frac{1}{2}$, we deduce from Proposition 30 that

$$\limsup_{n\to+\infty} \mathbf{E}\left[\Gamma[F_n, F_n]^{-q}\right] < +\infty, \quad q \in \mathbb{N}.$$

This shows $\mathcal{P}(m)$. This completes the induction step and thus the proof of Theorem 3. □

7. Multivariate random variables and sums of chaoses. In this section we prove Theorems 4 and 9.

7.1. *A central limit theorem for iterated sharp operators.* We recall that the iterated sharp operators are defined on polynomials $F \in \mathbb{R}[N_1, \dots, N_K]$ by

$$\sharp^k[F] := \sum_{1 \leq i_1, \dots, i_k} \frac{\partial^k F}{\partial N_{i_1} \cdots \partial N_{i_k}}(\vec{N}) G_{1,i_1} \cdots G_{k,i_k},$$

where $(G_{i,j})$ is a family of independent standard Gaussian independent of N_k. We prove that on Wiener chaoses, the property of converging to a Gaussian distribution is preserved by applications of iterated sharp.

PROPOSITION 41. *For any sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{W}_m$, we have*

$$F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1) \Rightarrow \sharp^k[F_n] \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = m(m-1)\cdots(m-k+1)$.

We prove the following lemma. We write $\gamma := \mathcal{N}(0,1)$ and $\gamma^{\mathbb{N}} := \bigotimes_{k\in\mathbb{N}}\gamma$.

LEMMA 42. *Let (F_n) a sequence in $\mathcal{W}_m^{(0)}$ such that $F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1)$. Then there exists a subsequence $(F_{\phi(n)})$ such that, for $\gamma^{\mathbb{N}}$-almost every sequence $(x_i)_i$,*

$$\sum_i \frac{\partial F_{\phi(n)}}{\partial N_i}(\vec{N})x_i \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,m).$$

PROOF. For an infinite vector $\vec{x} := (x_1, x_2, \ldots)$, we set

$$\mathsf{D}_{\vec{x}}F_n := \sum_i \frac{\partial F_n}{\partial N_i}(\vec{N})x_i,$$

where the sum is finite since $F_n \in \mathcal{W}_m^{(0)}$. By Proposition 35 with $d=1$, we deduce that the sequence of measurable mappings $\vec{x} \mapsto d_{\mathrm{FM}}(\mathsf{D}_{\vec{x}}F_n, \mathcal{N}(0,m))$ tends to 0 in probability on $(\mathbb{R}^{\mathbb{N}}, \gamma^{\mathbb{N}})$. Thus, there exists a subsequence which converges for $\gamma^{\mathbb{N}}$-almost every vector $\vec{x}$. The result follows. □

PROOF OF PROPOSITION 41. By successive applications of Lemma 42, there exists a subsequence $(F_{\phi(n)})$ such that, for $\gamma^{\mathbb{N}}\otimes\cdots\otimes\gamma^{\mathbb{N}}$-almost every sequences $(x_{1,i})_i, \ldots, (x_{k,i})_i$,

$$\sum_{1\le i_1,\ldots,i_k} \frac{\partial^k F_{\phi(n)}}{\partial N_{i_1}\cdots\partial N_{i_k}}(\vec{N})x_{1,i_1}\cdots x_{k,i_k} \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = m(m-1)\cdots(m-k+1)$. Take a continuous and bounded function $h\colon \mathbb{R}\to\mathbb{R}$. By the previous convergence, we find that, for $\gamma^{\mathbb{N}}\otimes\cdots\otimes\gamma^{\mathbb{N}}$-almost every sequences $(x_{1,i})_i, \ldots, (x_{k,i})_i$,

$$\mathbf{E}\left[h\left(\sum_{1\le i_1,\ldots,i_k} \frac{\partial^k F_{\phi(n)}}{\partial N_{i_1}\cdots\partial N_{i_k}}(\vec{N})x_{1,i_1}\cdots x_{k,i_k}\right)\right] \xrightarrow[n\to\infty]{} \int h(\sigma x)\gamma(\mathrm{d}x).$$

Integrating each of $x_{i,j}$'s with respect to $\gamma^{\mathbb{N}}$, we obtain, by dominated convergence, that

$$\mathbf{E}\big[h\big(\sharp^k[F_{\varphi(n)}]\big)\big] \xrightarrow[n\to\infty]{} \int h(\sigma x)\gamma(\mathrm{d}x).$$

This gives convergence in law of the subsequence. Since this reasoning applies on every subsequence of (F_n), we conclude on the convergence in law of the full sequence. □

7.2. *Proofs of the remaining theorems.* The following statement is slightly more precise than Theorem 9; we use it for the proof of Theorem 4. Recall that we write $\mathsf{J}_m F$ for the projection of F on the mth Wiener chaos.

PROPOSITION 43. *Let m and $q \in \mathbb{N}$. There exist $\delta > 0$, $r > 0$, and $C > 0$ such that the following statement holds: for every $F \in \mathcal{W}_{\le m}$,*

$$\big[d_{\mathrm{FM}}\big(\mathsf{J}_m F, \mathcal{N}(0,1)\big) \le \delta\big] \Rightarrow \big[\mathbf{E}\big[\Gamma[F,F]^{-q}\big] \le C\|F\|_{L^2}^r\big].$$

PROOF. Let $F \in \mathcal{W}_{\le m}$ and $\widetilde{F} := \mathsf{J}_m F$. By definition of the sharp operators, $\sharp^m[\widetilde{F}] = \sharp^m[F]$. Let $q \in \mathbb{N}$. Applying Proposition 29 to $\|F\|_{L^2}^{-1} F$, there exist $q' \in \mathbb{N}$ and $C > 0$ such that

$$\mathbf{E}\big[\Gamma[F, F]^{-q}\big] \le C \|F\|_{L^2}^{r} \,\mathbf{E}\big[\Gamma\big[\sharp^m[F], \sharp^m[F]\big]^{-q'}\big],$$

where $r := 2(q' - q)$. By Proposition 41, for every $\delta' > 0$, there exists $\delta > 0$ such that

$$d_{\mathrm{FM}}\big(\widetilde{F}, \mathcal{N}(0,1)\big) \le \delta \Rightarrow d_{\mathrm{FM}}\big(\sharp^m[\widetilde{F}], \mathcal{N}(0,\sigma^2)\big) \le \delta',$$

where $\sigma := m!^{\frac{1}{2}}$. By Theorem 3, for every $q' \in \mathbb{N}$, there exist $\delta' > 0$ and $C > 0$ such that

$$d_{\mathrm{FM}}\big(\sharp^m[F], \mathcal{N}(0,\sigma^2)\big) \le \delta' \Rightarrow \mathbf{E}\big[\Gamma\big[\sharp^m[F], \sharp^m[F]\big]^{-q'}\big] \le C.$$

Combining the three estimates above gives the result. □

Theorem 9 follows immediately. Now, we prove Theorem 4. In order to use Proposition 31, we prove the following lemma.

LEMMA 44. *Let $d \in \mathbb{N}^*$, $m_1, \dots, m_d \in \mathbb{N}^*$, and $q > 0$. There exist $\delta > 0$ and $C > 0$ such that the following statement holds: for every $\vec{F} = (F_1, \dots, F_d) \in \mathcal{W}_{m_1} \times \cdots \times \mathcal{W}_{m_d}$ such that $d_{\mathrm{FM}}(\vec{F}, \mathcal{N}(0, I_d)) \le \delta$, for every $\vec{a} \in \mathbb{S}^{d-1}$, the variable $F_{\vec{a}} = \sum_{i=1}^d a_i F_i = \vec{F} \cdot \vec{a}$ satisfies*

$$\mathbf{E}\big[\Gamma[F_{\vec{a}}, F_{\vec{a}}]^{-q}\big] \le C.$$

PROOF. We proceed by induction on d. The case $d = 1$ is Theorem 3. We fix $d \ge 2$, $m_1, \dots, m_d$, $F_1, \dots, F_d$, and $\vec{a}$ as in the statement. We set $m := \max_i m_i$. For $\epsilon > 0$, we bound $\mathbf{P}[\Gamma[F_{\vec{a}}, F_{\vec{a}}] \le \epsilon]$ in two different ways, according to the relative size of $|a_d|$ compared to ϵ. Fix $q \in \mathbb{N}$ and $\epsilon \in (0, 1/2)$, and set $\alpha := \min(\frac{q}{2r}, 1)$ where r is given by Proposition 43:

- Assume $|a_d| \ge \epsilon^\alpha$. Then Proposition 43, applied to $\frac{1}{a_d} F_{\vec{a}}$, implies that if $d_{\mathrm{FM}}(F_d, \mathcal{N}(0,1))$ is small enough, then

$$\mathbf{E}\big[\Gamma[F_{\vec{a}}, F_{\vec{a}}]^{-q}\big] \le C|a_d|^{-r}$$

  for a constant $C = C_{d,m,q}$. Thus,

$$\mathbf{P}\big[\Gamma[F_{\vec{a}}, F_{\vec{a}}] \le \epsilon\big] \le C|a_d|^{-r}\epsilon^q \le C\epsilon^{\frac{q}{2}}.$$

- Assume $|a_d| \le \epsilon^\alpha$. Define

$$F'_{\vec{a}} := \sum_{i=1}^{d-1} a_i F_i.$$

  By using the induction hypothesis with some number Q to be chosen later, we find a constant $C = C_{d,m,Q} > 0$ such that if $d_{\mathrm{FM}}(\vec{F}', \mathcal{N}(0, I_{d-1}))$ is small enough, then

$$\mathbf{E}\big[\Gamma\big[F'_{\vec{a}}, F'_{\vec{a}}\big]^{-Q}\big] \le C.$$

  By hypercontractivity (13) we find another constant $C = C_m > 0$ such that

$$\big\|\Gamma\big[F'_{\vec{a}}, F'_{\vec{a}}\big] - \Gamma[F_{\vec{a}}, F_{\vec{a}}]\big\|_{L^Q} \le C|a_d| \le C\epsilon^\alpha.$$

  Then, using that $\epsilon \le \frac{1}{2}\epsilon^{\frac{\alpha}{2}}$ (since $\alpha \le 1$ and $\epsilon < 1/2$), we write

$$\mathbf{P}\big[\Gamma[F_{\vec{a}}, F_{\vec{a}}] \le \epsilon\big] \le \mathbf{P}\big[\Gamma\big[F'_{\vec{a}}, F'_{\vec{a}}\big] \le \epsilon^{\frac{\alpha}{2}}\big] + \mathbf{P}\Big[\big|\Gamma\big[F'_{\vec{a}}, F'_{\vec{a}}\big] - \Gamma[F_{\vec{a}}, F_{\vec{a}}]\big| \ge \frac{1}{2}\epsilon^{\frac{\alpha}{2}}\Big],$$

where

$$\mathbf{P}\big[\Gamma[F'_{\vec{a}}, F'_{\vec{a}}] \le \epsilon^{\frac{\alpha}{2}}\big] \le C\epsilon^{Q\frac{\alpha}{2}}$$

$$\mathbf{P}\bigg[\big|\Gamma[F'_{\vec{a}}, F'_{\vec{a}}] - \Gamma[F_{\vec{a}}, F_{\vec{a}}]\big| \ge \frac{1}{2}\epsilon^{\frac{\alpha}{2}}\bigg] \le \bigg(\frac{2\|\Gamma[F'_{\vec{a}}, F'_{\vec{a}}] - \Gamma[F_{\vec{a}}, F_{\vec{a}}]\|_{L^Q}}{\epsilon^{\frac{\alpha}{2}}}\bigg)^Q \le C\epsilon^{Q\frac{\alpha}{2}}.$$

Choosing Q such that $Q\alpha \ge q$, for instance, $Q \ge 4r$, we deduce that there exists $C = C_{d,m,q}$ such that

$$\mathbf{P}\big[\Gamma[F_{\vec{a}}, F_{\vec{a}}] \le \epsilon\big] \le C\epsilon^{\frac{q}{2}}.$$

Combining the two cases, we obtained that for every $q \ge 0$ if $d_{\mathrm{FM}}(\vec{F}, \mathcal{N}(0, I_d))$ is small enough, there exists $C = C_{d,m,q}$ such that

$$\mathbf{P}\big[\Gamma[F_{\vec{a}}, F_{\vec{a}}] \le \epsilon\big] \le C\epsilon^{\frac{q}{2}}, \quad \varepsilon > 0.$$

The conclusion follows. $\square$

PROOF OF THEOREM 4. Let $m_1, \dots, m_d \in \mathbb{N}^*$. Consider a sequence $(\vec{F}_n) \subset \mathcal{W}_{m_1} \times \cdots \times \mathcal{W}_{m_d}$, converging in law to $\mathcal{N}(0, I_d)$, and a sequence $(\vec{a}_n) \subset \mathbb{S}^{d-1}$. For every $q \ge 0$, we apply Lemma 44 to $\vec{F}_n$, and we deduce that there exists $N \in \mathbb{N}$ such that

$$\sup_{n \ge N} \mathbf{E}\big[\Gamma[\vec{F}_n \cdot \vec{a}_n, \vec{F}_n \cdot \vec{a}_n]^{-q}\big] < +\infty, \quad n \ge N.$$

Then by Proposition 31, for every $q \ge 0$, there exists $N \in \mathbb{N}$ such that

$$\mathbf{E}\big[\det \Gamma(\vec{F}_n)^{-q}\big] \le C, \quad n \ge N. \qquad \square$$

Acknowledgments. The authors are grateful to G. Cébron for suggesting the reference [5] regarding superconvergence in the free central limit theorem. The authors are indebted to the anonymous referee. Their thorough review and thoughtful comments have greatly helped us in the preparation of the final version of this paper.

Funding. R.H. gratefully acknowledges funding from Centre Henri Lebesgue (ANR-11-LABX-0020-01) through a research fellowship in the framework of the France 2030 program.

This work was supported by the ANR Grant UNIRANDOM (ANR-17-CE40-0008).

REFERENCES

[1] ARTSTEIN, S., BALL, K. M., BARTHE, F. and NAOR, A. (2004). On the rate of convergence in the entropic central limit theorem. *Probab. Theory Related Fields* **129** 381–390. MR2128238 https://doi.org/10.1007/s00440-003-0329-4

[2] ARU, J., JEGO, A. and JUNNILA, J. (2022). Density of imaginary multiplicative chaos via Malliavin calculus. *Probab. Theory Related Fields* **184** 749–803. MR4507934 https://doi.org/10.1007/s00440-022-01135-y

[3] AZMOODEH, E., CAMPESE, S. and POLY, G. (2014). Fourth Moment Theorems for Markov diffusion generators. *J. Funct. Anal.* **266** 2341–2359. MR3150163 https://doi.org/10.1016/j.jfa.2013.10.014

[4] AZMOODEH, E., MALICET, D., MIJOULE, G. and POLY, G. (2016). Generalization of the Nualart–Peccati criterion. *Ann. Probab.* **44** 924–954. MR3474463 https://doi.org/10.1214/14-AOP992

[5] BERCOVICI, H. and VOICULESCU, D. (1995). Superconvergence to the central limit and failure of the Cramér theorem for free random variables. *Probab. Theory Related Fields* **103** 215–222. MR1355057 https://doi.org/10.1007/BF01204215

[6] BOBKOV, S. G., CHISTYAKOV, G. P. and GÖTZE, F. (2013). Rate of convergence and Edgeworth-type expansion in the entropic central limit theorem. *Ann. Probab.* **41** 2479–2512. MR3112923 https://doi.org/10.1214/12-AOP780

[7] BOGACHEV, V. I. (2022). Chebyshev–Hermite polynomials and distributions of polynomials in Gaussian random variables. *Theory Probab. Appl.* **66** 550–569. MR4466399

[8] BOGACHEV, V. I., KOSOV, E. D. and ZELENOV, G. I. (2018). Fractional smoothness of distributions of polynomials and a fractional analog of the Hardy–Landau–Littlewood inequality. *Trans. Amer. Math. Soc.* **370** 4401–4432. MR3811533 https://doi.org/10.1090/tran/7181

[9] BOULEAU, N. (2003). *Error Calculus for Finance and Physics*: *The Language of Dirichlet Forms*. *De Gruyter Expositions in Mathematics* **37**. de Gruyter, Berlin. MR2079474 https://doi.org/10.1515/9783110199291

[10] BREUER, P. and MAJOR, P. (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** 425–441. MR0716933 https://doi.org/10.1016/0047-259X(83)90019-2

[11] CARBERY, A. and WRIGHT, J. (2001). Distributional and L^q norm inequalities for polynomials over convex bodies in $\mathbb{R}^n$. *Math. Res. Lett.* **8** 233–248. MR1839474 https://doi.org/10.4310/MRL.2001.v8.n3.a1

[12] CASS, T. and FRIZ, P. (2010). Densities for rough differential equations under Hörmander's condition. *Ann. of Math.* (2) **171** 2115–2141. MR2680405 https://doi.org/10.4007/annals.2010.171.2115

[13] CHATTERJEE, S. (2009). Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields* **143** 1–40. MR2449121 https://doi.org/10.1007/s00440-007-0118-6

[14] CHEN, L. H. Y. and POLY, G. (2015). Stein's method, Malliavin calculus, Dirichlet forms and the fourth moment theorem. In *Festschrift Masatoshi Fukushima*. *Interdiscip. Math. Sci.* **17** 107–130. World Sci. Publ., Hackensack, NJ. MR3379337 https://doi.org/10.1142/9789814596534_0006

[15] GROSS, L. (1975). Logarithmic Sobolev inequalities. *Amer. J. Math.* **97** 1061–1083. MR0420249 https://doi.org/10.2307/2373688

[16] HAIRER, M. (2011). On Malliavin's proof of Hörmander's theorem. *Bull. Sci. Math.* **135** 650–666. MR2838095 https://doi.org/10.1016/j.bulsci.2011.07.007

[17] HERRY, R., MALICET, D. and POLY, G. (2023). Regularity of laws via Dirichlet forms—Application to quadratic forms in independent and identically distributed random variables.

[18] HOFFMAN, A. J. and WU, C. W. (2016). A simple proof of a generalized Cauchy–Binet theorem. *Amer. Math. Monthly* **123** 928–930. MR3570225 https://doi.org/10.4169/amer.math.monthly.123.9.928

[19] HÖRMANDER, L. (1967). Hypoelliptic second order differential equations. *Acta Math.* **119** 147–171. MR0222474 https://doi.org/10.1007/BF02392081

[20] HÖRMANDER, L. (1990). *The Analysis of Linear Partial Differential Operators*. *I*. *Distribution Theory and Fourier Analysis*, 2nd ed. *Springer Study Edition*. Springer, Berlin. MR1065136 https://doi.org/10.1007/978-3-642-61497-2

[21] HORN, R. A. and JOHNSON, C. R. (2013). *Matrix Analysis*, 2nd ed. Cambridge Univ. Press, Cambridge. MR2978290

[22] HU, Y., LU, F. and NUALART, D. (2014). Convergence of densities of some functionals of Gaussian processes. *J. Funct. Anal.* **266** 814–875. MR3132731 https://doi.org/10.1016/j.jfa.2013.09.024

[23] HU, Y., NUALART, D., TINDEL, S. and XU, F. (2015). Density convergence in the Breuer–Major theorem for Gaussian stationary sequences. *Bernoulli* **21** 2336–2350. MR3378469 https://doi.org/10.3150/14-BEJ646

[24] JOHANSSON, K. (1998). On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* **91** 151–204. MR1487983 https://doi.org/10.1215/S0012-7094-98-09108-6

[25] JOHNSON, O. and BARRON, A. (2004). Fisher information inequalities and the central limit theorem. *Probab. Theory Related Fields* **129** 391–409. MR2128239 https://doi.org/10.1007/s00440-004-0344-0

[26] LEDOUX, M. (2012). Chaos of a Markov operator and the fourth moment condition. *Ann. Probab.* **40** 2439–2459. MR3050508 https://doi.org/10.1214/11-AOP685

[27] LEDOUX, M., NOURDIN, I. and PECCATI, G. (2015). Stein's method, logarithmic Sobolev and transport inequalities. *Geom. Funct. Anal.* **25** 256–306. MR3320893 https://doi.org/10.1007/s00039-015-0312-0

[28] LIONS, P.-L. and TOSCANI, G. (1995). A strengthened central limit theorem for smooth densities. *J. Funct. Anal.* **129** 148–167. MR1322646 https://doi.org/10.1006/jfan.1995.1046

[29] MALICET, D., NOURDIN, I., PECCATI, G. and POLY, G. (2016). Squared chaotic random variables: New moment inequalities with applications. *J. Funct. Anal.* **270** 649–670. MR3425898 https://doi.org/10.1016/j.jfa.2015.10.013

[30] MALLIAVIN, P. (1978). Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations* (*Res. Inst. Math. Sci., Kyoto Univ., Kyoto*, 1976). *Wiley-Intersci. Publ.* 195–263. Wiley, New York–Chichester–Brisbane. MR0536013

[31] MOSSEL, E., O'DONNELL, R. and OLESZKIEWICZ, K. (2010). Noise stability of functions with low influences: Invariance and optimality. *Ann. of Math.* (2) **171** 295–341. MR2630040 https://doi.org/10.4007/annals.2010.171.295

[32] NOURDIN, I. Malliavin Stein Webpage. Accessed: 2024-01-11. https://sites.google.com/site/malliavinstein/home.

[33] NOURDIN, I. and NUALART, D. (2016). Fisher information and the fourth moment theorem. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 849–867. MR3498012 https://doi.org/10.1214/14-AIHP656

[34] NOURDIN, I., NUALART, D. and POLY, G. (2013). Absolute continuity and convergence of densities for random vectors on Wiener chaos. *Electron. J. Probab.* **18** no. 22, 19. MR3035750 https://doi.org/10.1214/EJP.v18-2181

[35] NOURDIN, I. and PECCATI, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. MR2520122 https://doi.org/10.1007/s00440-008-0162-x

[36] NOURDIN, I. and PECCATI, G. (2012). *Normal Approximations with Malliavin Calculus. From Stein's Method to Universality. Cambridge Tracts in Mathematics* **192**. Cambridge Univ. Press, Cambridge. MR2962301 https://doi.org/10.1017/CBO9781139084659

[37] NOURDIN, I., PECCATI, G. and REINERT, G. (2009). Second order Poincaré inequalities and CLTs on Wiener space. *J. Funct. Anal.* **257** 593–609. MR2527030 https://doi.org/10.1016/j.jfa.2008.12.017

[38] NOURDIN, I., PECCATI, G. and RÉVEILLAC, A. (2010). Multivariate normal approximation using Stein's method and Malliavin calculus. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 45–58. MR2641769 https://doi.org/10.1214/08-AIHP308

[39] NOURDIN, I., PECCATI, G. and SWAN, Y. (2014). Entropy and the fourth moment phenomenon. *J. Funct. Anal.* **266** 3170–3207. MR3158721 https://doi.org/10.1016/j.jfa.2013.09.017

[40] NOURDIN, I. and ROSIŃSKI, J. (2014). Asymptotic independence of multiple Wiener–Itô integrals and the resulting limit laws. *Ann. Probab.* **42** 497–526. MR3178465 https://doi.org/10.1214/12-AOP826

[41] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[42] NUALART, D. and ORTIZ-LATORRE, S. (2008). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Process. Appl.* **118** 614–628. MR2394845 https://doi.org/10.1016/j.spa.2007.05.004

[43] NUALART, D. and PECCATI, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. MR2118863 https://doi.org/10.1214/009117904000000621

[44] PECCATI, G. and TUDOR, C. A. (2005). Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII. Lecture Notes in Math.* **1857** 247–262. Springer, Berlin. MR2126978 https://doi.org/10.1007/978-3-540-31449-3_17

[45] SHIGEKAWA, I. (1980). Derivatives of Wiener functionals and absolute continuity of induced measures. *J. Math. Kyoto Univ.* **20** 263–289. MR0582167 https://doi.org/10.1215/kjm/1250522278

[46] WIGNER, E. P. (1955). Characteristic vectors of bordered matrices with infinite dimensions. *Ann. of Math.* (2) **62** 548–564. MR0077805 https://doi.org/10.2307/1970079