PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 152, Number 1, January 2024, Pages 403–409
https://doi.org/10.1090/proc/16448
Article electronically published on October 13, 2023

A SHORT PROOF OF A STRONG FORM OF THE THREE DIMENSIONAL GAUSSIAN PRODUCT INEQUALITY

RONAN HERRY, DOMINIQUE MALICET, AND GUILLAUME POLY

(Communicated by Amarjit Singh Budhiraja)

Abstract. We prove a strong form of the Gaussian product conjecture in dimension three. Our purely analytical proof simplifies previously known proofs based on combinatorial methods or computer-assisted methods, and allows us to solve the case of any triple of even positive integers which remained open so far.

1. Introduction and main result

Contribution to the Gaussian product conjecture. In this note, we prove Theorem 1.1.

Theorem 1.1. *Let (X_1, X_2, X_3) be centered real Gaussian vector, and $p_1, p_2, p_3 \in 2\mathbb{N}$. Then,*

$$\mathbb{E}[X_1^{p_1} X_2^{p_2} X_3^{p_3}] \geq \mathbb{E}[X_1^{p_1}]\,\mathbb{E}[X_2^{p_2}]\,\mathbb{E}[X_3^{p_3}], \tag{1}$$

with equality if and only if X_1, X_2, X_3 are independent.

Hence, our result completely solves the case $n = 3$ of a strong form of the celebrated *Gaussian product conjecture*. For short, let us introduce the following notation.

Definition 1.2. We say that $n \in \mathbb{N}$, and $p_1, \dots, p_n \in (0, \infty)$ satisfy the *Gaussian product inequality*, and we write $\mathbf{GPI}_n(p_1, \dots, p_n)$ provided for all real centered Gaussian vectors $(X_1, \dots, X_n)$:

$$\mathbb{E}\left[\prod_{i=1}^{n} |X_i|^{p_i}\right] \geq \prod_{i=1}^{n} \mathbb{E}[|X_i|^{p_i}],$$

with equality if and only if $X_1, \dots, X_n$ are independent. With a slight abuse of notation, we might also write $\mathrm{GPI}_n(0, p_2, \dots, p_n)$ instead of $\mathrm{GPI}_{n-1}(p_2, \dots, p_n)$.

Conjecture 1.3. *For all $n \in \mathbb{N}$, and all $p_1, \dots, p_n \in 2\mathbb{N}$, $\mathbf{GPI}_n(p_1, \dots, p_n)$ holds.*

Historically, the first instance of **GPI**-type inequalities goes back to [1, Thm. 3], where J. Arias-de-Reyna establishes the *complex* counterpart of the conjecture and uses it to solve the *complex polarization problem* in relation with long-standing problems regarding bounds on polynomials in several variables. The short and

Received by the editors November 29, 2022, and, in revised form, January 11, 2023.
2020 *Mathematics Subject Classification.* Primary 60G15; Secondary 39B62.
The first author gratefully acknowledges funding from Centre Henri Lebesgue (ANR-11-LABX-0020-01) through a research fellowship in the framework of the France 2030 program. This work was supported by the ANR grant UNIRANDOM, (ANR-17-CE40-0008).

elegant argument of [1] exploits the fact that the monomials form an orthogonal system of the L^2-space with respect to the standard n-dimensional *complex* Gaussian measure, and it cannot be adapted to the *real* case of the Gaussian product conjecture. Despite having received considerable attention since the seminal contribution of [1], the general case of the conjecture had, until now, remained wide open. The previous state of the art regarding the Gaussian product conjecture was the following.

Theorem 1.4. *The following cases of Conjecture* 1.3 *are known.*

(a) *For all* $p_1, p_2 \in 2\mathbb{N}$, $\mathbf{GPI}_2(p_1, p_2)$.
(b) ([3]) *For all* $n \in \mathbb{N}$, $\mathbf{GPI}_n(2, 2, \ldots, 2)$.
(c) ([8]) *For all* $p \in 2\mathbb{N}$, $\mathbf{GPI}_3(p, p, p)$.
(d) ([16]) *For all* $p \in 2\mathbb{N}$, $\mathbf{GPI}_3(p, 6, 4)$ *and* $\mathbf{GPI}_4(p, 2, 2, 2)$.
(e) ([13]) *For all* p *and* $q \in 2\mathbb{N}$, $\mathbf{GPI}_3(2, p, q)$.

The above results are obtained through sophisticated methods. In particular, [8] relies on a heavily combinatorial approach in connection with the theory of Gaussian hypergeometric functions; while [13, 16] is a computer-assisted method based on the SOS algorithm which provides an explicit expansion of a positive multivariate polynomial into a sum of squared quantities. On the contrary, our approach is purely analytical and combines an optimization procedure through the use of Lagrange multipliers with Gaussian analysis. Our contribution not only drastically simplifies the proof of the known cases in dimension three (Theorem 1.4(c) and (d)), but it also enables us to fully resolve the three dimensional case, that is to say for every choice of even integer exponents.

Other works related to Gaussian product inequalities. One of the most striking recent contributions regarding Gaussian inequalities is the following Royen's inequality for Gaussian correlations.

Theorem 1.5 ([12])**.** *Consider integers* $1 \leq k \leq n$ *and* $(X_1, \ldots, X_n)$ *centered Gaussian vectors. Then,*

$$\mathbb{P}\left[\max_{1 \leq i \leq n} |X_i| \leq 1\right] \geq \mathbb{P}\left[\max_{1 \leq i \leq k} |X_i| \leq 1\right] \mathbb{P}\left[\max_{k < i \leq n} |X_i| \leq 1\right].$$

More precisely, [12] actually establishes Theorem 1.5 in the setting of the multivariate Gamma distributions; only the Gaussian inequality is relevant for our discussion here. Despite its similarity with **GPI**-type inequalities, the beautiful techniques of [12] based on explicit computations for Laplace transform of the square of Gaussian yielding a monotonicity property for the quantity $\mathbb{P}[\max_i |X_i| \leq 1]$ along an interpolation of covariance matrices *cannot* be adapted to solve Conjecture 1.3. Let us also observe that in Theorem 1.5, one can iteratively factor out Gaussian vectors of smaller size. In analogy, with this property, [2] considers a variant of Conjecture 1.3 where one can iteratively factor out Gaussian vectors of smaller size in the inequality and calls it "the strong form of the Gaussian product inequality". They have established it for Gaussian vectors whose covariance matrices have nonnegative entries, but they have, in the very same paper, observed that for Gaussian vectors with arbitrary covariance matrices this stronger form of the conjecture is false.

Since it is known that $\mathbf{GPI}_n(p, p, \ldots, p)$ for all $n \in \mathbb{N}$ and all $p \in 2\mathbb{N}$ is sufficient to imply the *real* polarization conjecture, certain authors, such as [8], have dubbed the case $p_1 = \cdots = p_n$ of Conjecture 1.3 the "Gaussian product conjecture". This is why we say that our inequality is "a strong form" of the above conjecture to highlight that our method allows to consider any triplet of distinct even integers. We do not claim that this denomination is canonical.

Actually, due to the difficulty of the general case of the conjecture, several authors have tackled subcases of it by considering only Gaussian vectors whose covariance matrices have a particular form or other related distributions. Let us mention [2, 4, 7, 15] for such results with increasing degrees of generality related to some form of positivity. In the same spirit, [11] obtains a Gaussian product inequality for the case of multinomial covariances.

Other authors have also considered variations of Conjecture 1.3 where they consider other functions than monomials. For instance, [10] establishes a variant of the Gaussian product inequality involving *Hermite polynomials*; [17] proves $\mathbf{GPI}_n(p_1, \ldots, p_n)$ for all $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in (-1, 0)$; [9] establishes a version of the inequality involving trigonometric functions; [14] derives a reverse $\mathbf{GPI}_2(p_1, p_2)$ when $p_1 \in (-1, 0)$ and $p_2 > 0$. [5] later generalizes those results to the case of trace Wishart distributions and to completely monotone functions.

2. Proof of the main result

In the rest of the paper, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting an independent sequence $(G_k)_{k\in\mathbb{N}}$ of centered normalized Gaussian variables on Ω. In the following, $\Sigma = (\sigma_{i,j})$ is a real symmetric non-negative matrix of size n. To Σ, we associate a centered Gaussian vector $\vec{X} = (X_1, \ldots, X_n)$ with covariance matrix Σ by setting $\vec{X} = \Sigma^{1/2}\vec{G}$ where $\vec{G} = (G_1, \ldots, G_n)$. Let $p_1, \ldots, p_n \in 2\mathbb{N}^*$, and $h(x_1, \ldots, x_n) = x_1^{p_1} \ldots x_n^{p_n}$. Our strategy consists in studying the points where the map $\Phi\colon \Sigma \mapsto \mathbb{E}[h(X_1, \ldots, X_n)]$ reaches its minimum. Our argument allows us to characterize those minimal points for $n = 2$ or 3. Using Wick formula [6, Thm. 1.28], it is readily checked that Φ is polynomial in the entries of Σ. We shall need the following standard lemma. We recall a proof for the sake of self-containedness.

Lemma 2.1. *Let $(X_1, \cdots, X_n)$ be a Gaussian vector that is centered with covariance matrix Σ non-necessarily invertible. Then it holds:*

$$\mathbb{E}[X_i h(X_1, \ldots, X_n)] = \sum_{j=1}^{n} \sigma_{i,j}\, \mathbb{E}\big[\partial_{x_j} h(X_1, \ldots, X_n)\big], \qquad i \in \{1, \ldots, n\}; \tag{2}$$

$$\frac{\partial}{\partial \sigma_{i,j}}\, \mathbb{E}[h(X_1, \ldots, X_n)] = \mathbb{E}\big[\partial_{x_i}\partial_{x_j} h(X_1, \ldots, X_n)\big], \qquad i \neq j \in \{1, \ldots, n\}. \tag{3}$$

Proof of Lemma 2.1. In view of Wick formula, (2) and (3) are equalities involving polynomials in the entries of Σ. It is thus sufficient to establish them for an invertible Σ. In this case, let us write f_Σ for the density distribution associated with $\vec{X}$. A direct computation yields

$$x_i f_\Sigma + \sum_{j=1}^{n} \sigma_{i,j}\partial_{x_j} f_\Sigma = 0, \qquad i \in \{1, \ldots, n\}.$$

(2) readily follows. In order to prove (3), consider the Fourier transform of f_Σ:

$$\widehat{f_\Sigma}(x_1, \cdots, x_n) = \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} x_i x_j \sigma_{i,j}\right), \qquad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Differentiating this formula, we obtain for $i \neq j$:

$$\widehat{\partial_{\sigma_{i,j}} f_\Sigma} = \partial_{\sigma_{i,j}} \widehat{f_\Sigma} = -x_i x_j \widehat{f_\Sigma} = \widehat{\partial_{x_i} \partial_{x_j} f_\Sigma}.$$

Since the Fourier transform is into, we get that $\partial_{\sigma_{i,j}} f_\Sigma = \partial_{x_i}\partial_{x_j} f_\Sigma$, from which (2) readily follows. □

In order to highlight the line of reasoning we use in the proof of Theorem 1.1, and for the sake of completeness, let us first give a short proof of Theorem 1.4(a).

Proof of Theorem 1.4(a). Fix p_1 and $p_2 \in 2\mathbb{N}^*$. We want to prove that if (X_1, X_2) is a centered real Gaussian vector then $\mathbb{E}[X_1^{p_1} X_2^{p_2}] \geq \mathbb{E}[X_1^{p_1}]\,\mathbb{E}[X_2^{p_2}]$, with equality if and only if X_1 and X_2 are independent. By homogeneity it is enough to prove the statement when X_1 and X_2 are normalized, in which case the term on the right-hand side of the inequality depends only on p_1 and p_2. Setting, for t in $[-1, 1]$, $\Phi(t) = \mathbb{E}[X_1^{p_1} X_2^{p_2}]$ where (X_1, X_2) is the Gaussian vector associated to

$$\Sigma = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix},$$

the claim is equivalent to the fact that Φ reaches its unique minimum at $t = 0$. From (3), $\Phi'(t) = p_1 p_2\, \mathbb{E}[X_1^{p_1-1} X_2^{p_2-1}]$ and $\Phi''(t) = p_1(p_1-1)p_2(p_2-1)\,\mathbb{E}[X_1^{p_1-2} X_2^{p_2-2}]$. In particular, $\Phi'' > 0$ and $\Phi'(0) = 0$. Consequently, 0 is a critical point of a strictly convex function, and thus it is the unique global minimizer of Φ, from which the result follows. □

Theorem 1.1 follows from the recursive argument below; the corresponding initialization is given by Theorem 1.4(a).

Proposition 2.2. *For all p_1, p_2, and $p_3 \in 2\mathbb{N}^*$,*

$$\mathbf{GPI}_3(p_1 - 2, p_2, p_3) \Rightarrow \mathbf{GPI}_3(p_1, p_2, p_3).$$

Proof. Let $\mathcal{C}$ be the set of real symmetric positive matrices of size 3 with 1 on the diagonal, namely

$$\mathcal{C} = \left\{ \Sigma = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ c & b & 1 \end{pmatrix} : |a|, |b|, |c| \leq 1,\ \det(\Sigma) \geq 0 \right\}.$$

We identify $\mathcal{C}$ with a compact subset of $\mathbb{R}^3$. With this notation, $\mathbf{GPI}_3(p_1, p_2, p_3)$ turns out to be equivalent to the fact that Φ attains its unique minimum on $\mathcal{C}$ at I_3. Since $\mathcal{C}$ is compact and Φ continuous, Φ has a global minimum at some possibly non-unique

$$\Sigma_0 = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \in \mathcal{C}.$$

We prove that $\Sigma_0 = I_3$. We split the argument in three cases, depending on the location of Σ_0 in $\mathcal{C}$.

Case 1. We assume that Σ_0 is in the interior on $\mathcal{C}$. This means that $\det(\Sigma_0) > 0$ and $|a|, |b|, |c| < 1$. In this case, Σ_0 is a critical point of Φ. Write

$$U = X_1^{p_1-1} X_2^{p_2-1} X_3^{p_3-1}.$$

According to (3)

$$\begin{cases} \partial_a \Phi(\Sigma_0) = p_1 p_2 \, \mathbb{E}[X_3 U], \\ \partial_b \Phi(\Sigma_0) = p_1 p_3 \, \mathbb{E}[X_2 U], \\ \partial_c \Phi(\Sigma_0) = p_2 p_3 \, \mathbb{E}[X_1 U]. \end{cases} \tag{4}$$

Thus, $\mathbb{E}[X_1 U] = \mathbb{E}[X_2 U] = \mathbb{E}[X_3 U] = 0$. On the other hand, let

$$V = X_1^{p_1 - 1} X_2^{p_2} X_3^{p_3}.$$

Thus, by (2) and the fact that the derivatives vanish,

$$\begin{aligned} \Phi(\Sigma_0) &= \mathbb{E}[X_1 V] \\ &= (p_1 - 1)\, \mathbb{E}[X_1^{p_1-2} X_2^{p_2} X_3^{p_3}] + p_2 a \, \mathbb{E}[X_3 U] + p_3 b \, \mathbb{E}[X_2 U] \\ &= (p_1 - 1)\, \mathbb{E}[X_1^{p_1-2} X_2^{p_2} X_3^{p_3}]. \end{aligned}$$

In view of $\mathbf{GPI}_3(p_1 - 2, p_2, p_3)$, we thus get

$$\Phi(\Sigma_0) \geq (p_1 - 1)\, \mathbb{E}[X_1^{p_1-2}]\, \mathbb{E}[X_2^{p_2}]\, \mathbb{E}[X_3^{p_3}] = \mathbb{E}[X_1^{p_1}]\, \mathbb{E}[X_2^{p_2}]\, \mathbb{E}[X_3^{p_3}] = \Phi(I_3).$$

Since Σ_0 is a minimizer, we actually have that $\Phi(\Sigma_0) = \Phi(I_3)$. In particular, this means that we are in the equality case of $\mathbf{GPI}_3(p_1 - 2, p_2, p_3)$. If $p_1 > 2$, this shows immediately that $\Sigma_0 = I_3$. Similarly, if $p_2 > 2$ or $p_3 > 2$, we conclude in the same way. If $p_1 = p_2 = p_3 = 2$, using Theorem 1.4(a), we deduce that the components of (X_1, X_2, X_3) are pairwise independent. Since the vector is Gaussian, the conclusion follows.

Case 2. We assume that $|a|, |b|, |c| < 1$ and $\det(\Sigma_0) = 0$.

Σ_0 is a priori not a critical point of Φ. Since Σ_0 is a global minimizer on $\mathcal{C}$ it is also a minimizer of Φ on the surface

$$S = \left\{ \Sigma = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} : \det(\Sigma) = 0 \right\}.$$

By the Lagrange multiplier theorem, we conclude that $\vec{\nabla}\Phi(\Sigma_0)$ and $\vec{\nabla}(\det)(\Sigma_0)$ are colinear (where $\vec{\nabla} = (\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c})$). We have already computed $\vec{\nabla}\Phi(\Sigma_0)$ in (4), and we have:

Lemma 2.3. $\vec{\nabla}(\det)(\Sigma_0) = (\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_2\alpha_3)$, *where* $(\alpha_1, \alpha_2, \alpha_3)$ *is some non-zero vector of* $\ker(\Sigma_0)$.

Proof. Write $A = 2\mathrm{adj}(\Sigma_0)$ where adj stands for the adjugate matrix. Since $\det(\Sigma_0) = 0$, $\mathrm{rank}(\Sigma_0) \leq 2$, and since $|a|, |b|, |c| < 1$, two columns of Σ_0 cannot be proportional so $\mathrm{rank}(\Sigma_0) = 2$. This implies that A has rank 1, thus $A = \alpha^T \alpha$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \ker(\Sigma_0) \setminus \{0\}$. By Jacobi's formula, we have that $\vec{\nabla}(\det)(\Sigma_0) = (A_{1,2}, A_{1,3}, A_{2,3})$. □

We deduce that there exists a real number k such that

$$\begin{cases} \partial_a \Phi(\Sigma_0) = k\alpha_1\alpha_2, \\ \partial_b \Phi(\Sigma_0) = k\alpha_1\alpha_3, \\ \partial_c \Phi(\Sigma_0) = k\alpha_2\alpha_3. \end{cases} \tag{5}$$

Since $(\alpha_1, \alpha_2, \alpha_3)$ belongs to $\ker(\Sigma_0)$, $\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0$ almost surely. From (4),

$$\begin{aligned} 0 &= p_1 p_2 p_3 \, \mathbb{E}[U(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3)] \\ &= p_1 \alpha_1 \frac{\partial \Phi}{\partial c}(\Sigma_0) + p_2 \alpha_2 \frac{\partial \Phi}{\partial b}(\Sigma_0) + p_3 \alpha_3 \frac{\partial \Phi}{\partial a}(\Sigma_0). \end{aligned}$$

Thus by reporting in (5),

$$0 = (p_1 + p_2 + p_3) k \alpha_1 \alpha_2 \alpha_3.$$

If $k = 0$, then (5) gives that Σ_0 is a critical point of Φ, and as in Case 1 we obtain that $\Sigma_0 = I_3$, which contradicts $\det(\Sigma_0) = 0$. If one of the α_i is zero, say α_1, then $\alpha_2 X_2 + \alpha_3 X_3 = 0$, so X_3 and X_2 are proportional, and since they are normalized, $X_3 = \pm X_2$, which contradicts $|c| < 1$. Hence, Case 2 cannot happen.

Case 3. We assume that $\{|a|, |b|, |c|\} \cap \{1\} \neq \emptyset$. Say for example that $|c| = 1$. That implies that $X_3 = \pm X_2$ and so by the two dimensional case Theorem 1.4(a),

$$\begin{aligned} \Phi(\Sigma_0) &= \mathbb{E}[{X_1}^{p_1} {X_2}^{p_2 + p_3}] \\ &\geq \mathbb{E}[{X_1}^{p_1}] \, \mathbb{E}[{X_2}^{p_2 + p_3}] \\ &> \mathbb{E}[{X_1}^{p_1}] \, \mathbb{E}[{X_2}^{p_2}] \, \mathbb{E}[{X_2}^{p_3}] = \Phi(I_3). \end{aligned}$$

In particular Σ_0 is not a minimizer. It is a contradiction, and Case 3 cannot either happen.

Conclusion. We obtain that the only minimizer of Φ is I_3 which concludes the proof. □

Remark 2.4 (Discussion on possible extensions to higher dimension). Our method could *theoretically* be applied to higher dimension. However, this extension presents additional difficulties that would require novel ideas that are outside of the scope of this short note. Let us briefly mention the main difficulty. For an arbitrary $n \in \mathbb{N}$, the boundary of set $\mathcal{C}$ is a union of hypersurfaces $(S_k)_{1 \leq k < n}$ of covariance matrices of rank k. On S_{n-1}, equations similar to (4) and (5) can be derived when considering a Gaussian vector of size $n \in \mathbb{N}$. However, the equations one obtains in this case are of degree $n - 2$ in the X_i's and we do not know how to exploit them in order to derive meaningful conditions on the α_i's or on the Lagrange multiplier. For $1 < k < n - 1$, on S_k the Lagrange multipliers theorem would take an even more complicated form.

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IRMAR, UMR CNRS 6625, Université de Rennes, F-35000 Rennes, France
Email address: ronan.herry@univ-rennes1.fr

LAMA, UMR CNRS 8050, Université Gustave Eiffel, F-77420 Champs-sur-Marne, France
Email address: dominique.malicet@univ-eiffel.fr

IRMAR, UMR CNRS 6625, Université de Rennes, F-35000 Rennes, France
Email address: guillaume.poly@univ-rennes1.fr