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RESEARCH ARTICLE

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Conformally invariant random fields, Liouville quantum gravity measures, and random Paneitz operators on Riemannian manifolds of even dimension

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Abstract

For large classes of *even-dimensional* Riemannian manifolds (M, g), we construct and analyze conformally invariant random fields. These centered Gaussian fields $h = h_g$, called *co-polyharmonic Gaussian fields*, are characterized by their covariance kernels k which exhibit a precise logarithmic divergence: $\left|k(x,y) - \log \frac{1}{d(x,y)}\right| \leq C$. They share a fundamental quasi-invariance property under conformal transformations. In terms of the co-polyharmonic Gaussian field h, we define the *Liouville Quantum Gravity measure*, a random measure on M, heuristically given as

$$d\mu_g^{\gamma h}(x) := e^{\gamma h(x) - \frac{\gamma^2}{2} k(x,x)} \, d\mathrm{vol}_g(x),$$

and rigorously obtained as almost sure weak limit of the right-hand side with h replaced by suitable regular approximations $h_\ell, \ell \in \mathbb{N}$. In terms on the Liouville Quantum Gravity measure, we define the *Liouville Brownian motion* on M and the *random GJMS operators*.

Finally, we present an approach to a conformal field theory in arbitrary even dimension with an ansatz based on Branson's Q-curvature: we give a rigorous meaning to the *Polyakov–Liouville measure*

$$d\boldsymbol{\nu}^*_g(h) = \frac{1}{Z^*_g} \exp\left(-\int \Theta\, Q_g h + m e^{\gamma h} \mathrm{dvol}_g\right) \times \exp\left(-\frac{a_n}{2} \mathfrak{p}_g(h,h)\right) dh$$

and we derive the corresponding *conformal anomaly*. The set of *admissible* manifolds is conformally invariant. It includes all compact 2-dimensional Riemannian manifolds, all compact non-negatively curved Einstein manifolds of even dimension, and large classes of compact hyperbolic manifolds of even dimension. However, not every compact even-dimensional Riemannian manifold is admissible. Our results concerning the logarithmic divergence of the kernel k rely on new sharp estimates for heat kernels and higher order Green kernels on arbitrary closed manifolds.

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Contents

INTRODUCTION

Conformally invariant random objects on the complex plane or on Riemannian surfaces are a central topic of current research and play a fundamental role in many mathematical theories. The last two decades have seen an impressive wave of fascinating constructions, deep insights, and spectacular results for various conformally (quasi-) invariant random objects, most prominently the Gaussian free field, the Liouville quantum gravity (LQG) measure, the Brownian map, and the SLE curves.

In this paper, we use ideas from conformal geometry in higher dimension to establish the foundations for a mathematical theory of conformally invariant random fields and LQG on compact Riemannian manifolds of even dimension.

Co-polyharmonic Gaussian fields

We construct conformally quasi-invariant random Gaussian fields h on admissible Riemannian manifolds (M, g) of arbitrary even dimension. The covariance kernels of these centered Gaussian fields, naively interpreted as $k_g(x, y) = \mathbf{E}\big[h(x)\, h(y)\big]$, exhibits a logarithmic divergence

$$\left| k_g(x, y) - \log \frac{1}{d(x, y)} \right| \leqslant C \,. \tag{1}$$

As for the Gaussian free field, these random fields, called *copolyharmonic Gaussian fields*, are not classical functions on M but rather elements in the Sobolev space $H_g^{-s} := H^{-s}(M, g)$ for any $s > 0$. By construction, they annihilate constants, that is, $\langle h \,|\, \mathbf{1} \rangle_g = 0$, where $\langle \cdot \mid \cdot \rangle_g$ denotes the pairing between H_g^{-s} and H_g^{s}.

We prove (Theorem 3.13) that co-polyharmonic Gaussian fields are conformally quasi-invariant: let h_g denote the co-polyharmonic Gaussian field for (M, g) and $h_{g'}$ that for (M, g') with $g' = e^{2\varphi} g$ and φ smooth, then

$$h_{g'} \overset{(\mathrm{d})}{=} h_g - C,$$

where C is an appropriate random variable that ensures that the right-hand side annihilates constants. To get rid of this additive correction term, one can consider the "random variable" $\hbar_g = h_g + a$, called *ungrounded co-polyharmonic Gaussian field* with h_g as above and a distributed according to the Lebesgue measure on $\mathbb{R}$. Then,

$$\hbar_{g'} \overset{(\mathrm{d})}{=} \hbar_g. \tag{2}$$

Here and in all the paper, we use $\overset{(\mathrm{d})}{=}$ to indicate that two random variables have the same law.

Co-polyharmonic operators

For a given closed manifold (M, g) of even dimension n, the covariance kernel k_g is the integral kernel for the inverse of the operator $\mathsf{p}_g := a_n \mathsf{P}_g$ on the "grounded" L^2-space $\mathring{L}^2_g := \{u \in L^2(M, \mathsf{vol}_g) : \int u \,d\mathsf{vol}_g = 0\}$. Here, $a_n := 2(4\pi)^{-n/2}/\Gamma(n/2)$ and

$$\mathsf{P}_g = (-\Delta_g)^{n/2} + \text{low order terms} \tag{3}$$

denotes the *co-polyharmonic operator* or *Graham–Jenne–Mason–Sparling operator of maximal order (GJMS operator for short).* The operator P_g plays the role of a conformally invariant power of the Laplacian and has been first defined in [40]. For $n = 2$, the nonnegative operator P_g is just $-\Delta_g$, the negative of the Laplacian, and for $n = 4$, it is the celebrated *Paneitz operator* [70].

The co-polyharmonic Gaussian field h on (M, g) can easily be constructed in terms of the eigenbasis $(\psi_j)_{j\in\mathbb{N}_0}$ of p_g: with $(\nu_j)_{j\in\mathbb{N}_0}$ the corresponding eigenvalues and any sequence $(\xi_j)_{j\in\mathbb{N}}$ of independent standard normal random variables, then (Proposition 3.9)

$$h = \lim_{\ell\to\infty} h_\ell\,, \qquad h_\ell(x) := \sum_{j=1}^{\ell} \frac{1}{\sqrt{\nu_j}} \psi_j(x)\, \xi_j\,, \tag{4}$$

with convergence in quadratic mean.

Liouville quantum gravity measures

We then define the *plain Liouville Quantum Gravity measure* $\mu_g^{\gamma h}$ on M for every parameter $\gamma \in \mathbb{R}$ with $|\gamma| < \sqrt{2n}$ as a random finite measure. Employing Kahane's idea of *Gaussian multiplicative chaos* (see [54]), we define (Theorem 4.1) the measure $\mu_g^{\gamma h}$ as the almost sure limit (in the usual sense of weak convergence of measures) of the sequence $(\mu_g^{\gamma h_\ell})_{\ell\in\mathbb{N}}$ of finite measures on M given by

$$d\mu_g^{\gamma h_\ell}(x) := e^{\gamma h_\ell(x) - \frac{\gamma^2}{2} k_\ell(x,x)}\, \mathrm{dvol}_g(x),$$

with h_ℓ as in (4) and $k_\ell(x,x) := \mathbf{E}[h_\ell(x)^2] = \sum_{j=1}^{\ell} \frac{1}{\nu_j}\psi_j(x)^2$.

We establish that almost surely the measure $\mu_g^{\gamma h}$ is a finite measure on M with full topological support, and for every $s > \gamma^2/4$, it does not charge sets of vanishing H^s-capacity (Theorem 5.2). In particular, it does not charge sets of vanishing $H^{n/2}$-capacity since $|\gamma| < \sqrt{2n}$ throughout. If, moreover, $|\gamma| < 2$ then, almost surely, $\mu_g^{\gamma h}$ does not charge sets of vanishing H^1-capacity.

The plain LQG measure has the following crucial quasi-invariance property (Theorem 4.15): if $g' = e^{2\varphi} g$, then

$$\mu_{g'}^{\gamma h'} \overset{(\mathrm{d})}{=} e^F \mu_g^{\gamma h}, \tag{5}$$

where F is the random variable explicitly given in (112). We also study the *("adjusted") LQG measures*, denoted by $\bar\mu$, which is equal, up to a deterministic multiplicative weight to the plain LQG measure defined above, shares many properties with it, and exhibits a simpler quasi-invariance property (Theorem 4.20):

$$\bar\mu_{g'}^{\gamma h'} \overset{(\mathrm{d})}{=} e^{-\gamma\xi}\, e^{(n+\gamma^2/2)\varphi}\, \bar\mu_g^{\gamma h} ,$$

where $\xi := \langle h \rangle_{g'}$ is a normal random variable. Passing from the grounded to the ungrounded co-polyharmonic field, this finally reads as

$$\bar\mu_{g'}^{\gamma \hbar'} \overset{(\mathrm{d})}{=} e^{(n+\gamma^2/2)\varphi}\, \bar\mu_g^{\gamma \hbar} . \tag{6}$$

Random quadratic forms

With respect to the LQG measure, we can define a variety of random objects that play a fundamental role in geometric analysis, spectral theory, and probabilistic potential theory.

Restricting to the range $\gamma \in (-2, 2)$, we construct (Theorem 5.7) a *random Dirichlet form* on $L^2(M, \mu_g^{\gamma h})$ by:

$$\mathcal{E}_g^h(u,u) := \int_M |\nabla u|^2\, \mathrm{dvol}_g, \qquad \mathcal{D}(\mathcal{E}_g^h) := H^1(M) \cap L^2\left(M, \mu_g^{\gamma h}\right) .$$

The associated reversible and continuous Markov process is the *Liouville Brownian motion* (see [45, 46] and [6] for two independent constructions on the plane). It is obtained from the standard Brownian motion on (M, g) through time change. The new time scale is given as the right inverse of the additive functional

$$A_t^h = \lim_{\ell\to\infty} \int_0^t \exp\left(\gamma\, h_\ell(X_s) - \frac{\gamma^2}{2} k_\ell(X_s, X_s)\right) ds .$$

In dimension $n > 2$, however, this Liouville Brownian motion has no canonical invariance property under conformal transformations simply because its generator is not conformally invariant.

To obtain conformally quasi-invariant random objects in higher dimensions, our starting point, in Theorem 5.12, is the random co-polyharmonic form

$$\mathfrak{E}_g^h(u,v) := \int_M u\, \mathsf{P}_g v\, d\mathrm{vol}_g\ , \qquad \mathcal{D}(\mathfrak{E}_g^h) := H^{n/2}(M) \cap L^2\Big(M, \mu_g^{\gamma h}\Big)$$

(rather than the random Dirichlet form) which in the full range $\gamma \in (-\sqrt{2n}, \sqrt{2n})$ is, almost surely, a well-defined nonnegative closed symmetric bilinear form on $L^2(M, \mu_g^{\gamma h})$. It allows us to define *random co-polyharmonic operators* P_g^h. The associated random co-polyharmonic heat flow $e^{-t\mathsf{P}_g^h}$ is the gradient flow for the quadratic functional $\frac{1}{2}\mathfrak{E}_g^h$ in the random landscape $L^2(M, \mu_g^{\gamma h})$ (Proposition 5.15).

In Theorem 5.17, we show that the random co-polyharmonic operators share the fundamental quasi-invariance property

$$\mathsf{P}_{g'}^{h'} \overset{(\mathrm{d})}{=} e^{-F}\, \mathsf{P}_g^h\ ,$$

with F as in (5).

Polyakov–Liouville measure

Finally, we propose an ansatz for a Liouville conformal field theory on closed manifolds of arbitrary even dimension. Our approach, based on Branson's Q-curvature, provides a rigorous meaning to the Polyakov–Liouville measure $\boldsymbol{\pi}_g$, informally given as

$$\boldsymbol{\pi}_g(dh) = \exp(-S_g(h))dh$$

with the (nonexisting) uniform distribution dh on the set of fields and the action, heuristically considered first in [60, Eqn. (1.1)],

$$S_g(h) := \int_M \Big(\tfrac{1}{2}\, h\mathsf{p}_g h + \Theta\, Q_g h + m e^{\gamma h}\Big) d\mathrm{vol}_g\ ,$$

where $m, \Theta, \gamma > 0$ are parameters (subjected to some restrictions). To rigorously introduce the *("adjusted") Polyakov–Liouville measure* $\bar{\boldsymbol{\pi}}_g$, we define it as

$$\bar{\boldsymbol{\pi}}_g := \sqrt{\frac{\mathrm{vol}_g(M)}{\det'\Big(\frac{1}{2\pi}\mathsf{p}_g\Big)}} \cdot \bar{\boldsymbol{\nu}}_g^*$$

with

$$d\bar{\boldsymbol{\nu}}_g^*(h+a) := \exp\Big(-\Theta\big\langle h+a \,\big|\, Q_g\big\rangle_g - m\, e^{\gamma a} \bar{\mu}_g^{\gamma h}(M)\Big)\, da\, d\boldsymbol{\nu}_g(h)\ , \tag{7}$$

where $\boldsymbol{\nu}_g$ denotes the law of the co-polyharmonic Gaussian field, informally understood as $d\boldsymbol{\nu}_g(h) = \frac{1}{Z_g}\exp(-\frac{1}{2}\langle h \,|\, \mathsf{p}_g h\rangle)\, dh$ with $Z_g := \sqrt{\mathrm{vol}_g(M)/\det'(\frac{1}{2\pi}\mathsf{p}_g)}$, and where $\bar{\mu}_g^{\gamma h}$ denotes the adjusted LQG measure. We prove (Theorem 6.11) that for admissible manifolds of negative total

Q-curvature, the measures $\bar{\boldsymbol{\nu}}^*_g$ and $\bar{\boldsymbol{\pi}}_g$ are finite. Moreover, for the particular choice $\Theta := a_n(\frac{n}{\gamma}+\frac{\gamma}{2})$, the measure $\bar{\boldsymbol{\nu}}^*_g$ is *quasi-invariant modulo shift* with $T:\ h \mapsto h-(\frac{n}{\gamma}+\frac{\gamma}{2})\varphi$ and *A-type conformal anomaly*

$$\frac{d\bar{\boldsymbol{\nu}}^*_{g'}}{dT_*\bar{\boldsymbol{\nu}}^*_g} = \exp\left(\left(\frac{n}{\gamma}+\frac{\gamma}{2}\right)^2\left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi)+a_n\int_M \varphi\, Q_g\, \mathrm{dvol}_g\right]\right) \tag{8}$$

(Theorem 6.12). In dimensions 2 and 4, we also determine the *"full" conformal anomaly* for transformations of the Polyakov–Liouville measure $\bar{\boldsymbol{\pi}}_g$, confirming the result from [47] in the case $n=2$ and providing a new formula in the case $n=4$:

$$\frac{d\bar{\boldsymbol{\pi}}_{g'}}{dT_*\bar{\boldsymbol{\pi}}_g} = \exp\left(\left[\frac{7}{45}+\left(\frac{n}{\gamma}+\frac{\gamma}{2}\right)^2\right]\cdot\left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi)+a_n\int \varphi\, Q_g\, \mathrm{dvol}_g\right]\right)$$
$$\cdot\exp\left(\frac{1}{45\pi^2}\left[-\int \mathrm{scal}^2_{g'} \mathrm{dvol}_{g'}+\int \mathrm{scal}^2_g \mathrm{dvol}_g\right]\right)\cdot\exp\left(\frac{1}{1440\pi^2}\int \varphi|W|^2 \mathrm{dvol}_g\right).$$

Admissible manifolds

Co-polyharmonic Gaussian fields do *not* exist on every Riemannian manifold. A closed (i.e., compact and without boundary) even-dimensional Riemannian manifold (M,g) is called *admissible* if $\mathsf{P}_g>0$ on $\mathring{L}^2_g$. Admissibility is a conformal invariance. All closed, nonnegatively curved Einstein manifolds are admissible, and so are all closed hyperbolic manifolds with spectral gap $\lambda_1 > \frac{n(n-2)}{4}$. Of course, all closed two-dimensional Riemannian manifolds are admissible.

We prove (Theorem 2.19, see also [66, Lemma 2.1]) that for every admissible manifold, the inverse of $\mathsf{p}_g := a_n \mathsf{P}_g$ on $\mathring{L}^2_g$ has an integral kernel k_g that annihilates constants and satisfies

$$\left|k_g(x,y)-\log\frac{1}{d(x,y)}\right| \leqslant C\,. \tag{9}$$

The two-dimensional case

Even in the case of surfaces, our approach provides new insights for the study of two-dimensional random objects. It applies to closed Riemannian surfaces of arbitrary genus, and thus, some of our results are new in the two-dimensional setting. In particular, we present a detailed discussion of the difference between plain and adjusted LQG measures as well as novel conformal transformation formulas for the plain LQG measure (Theorem 4.20, Cor. 4.16, 4.17). Moreover, considering the plain LQG measure (rather than the adjusted one) proves crucial for obtaining the new result on eigenbasis approximation

$$\mu_g^{\gamma h_\ell} \to \mu_g^{\gamma h} \tag{10}$$

(Theorem 4.14).

In addition to these novel results, our approach recovers many of the famous results concerning the Gaussian free field and the associated LQG measure in dimension 2, and for the first time, it provides an *intrinsic* Riemannian, conformally quasi-invariant extension to higher dimensions.

Probabilistic context

In dimension 2, conformally invariant random objects appear naturally in the study of continuum statistical models. The celebrated *Gaussian free field* arises as the scaling limit of various discrete models of random surfaces, for instance, discrete Gaussian free fields or harmonic crystals [79]. A planar conformally invariant random object of fundamental importance is the *Schramm–Loewner evolution* [55, 56, 77]. It plays a central role in many problems in statistical physics and satisfies some conformal invariance. The Schramm–Loewner evolution and the two-dimensional Gaussian free field are deeply related. For instance, level curves of the discrete Gaussian free field converge to SLE_4 [80], and zero contour lines of the Gaussian free field are well-defined random curves distributed according to SLE_4 [81]. The work [64], and subsequent works in its series, thoroughly study the relation between the Schramm–Loewner evolution and Gaussian free field on the plane. Motivated by Polyakov's informal formulation of Bosonic string theory [73, 74], the papers [28, 34, 47] construct mathematically the *LQG* on some surfaces and study its conformal invariance properties. Formally speaking, the LQG is a random surface obtained by random conformal transform of the Euclidean metric, where the conformal weight is the Gaussian free field. Since the Gaussian free field is only a distribution, we do not obtain a random Riemannian manifold but rather a random metric measure space. The aforementioned works construct the random measure based on a renormalization procedure due to Kahane [54]. This renormalization depends on a roughness parameter γ and works only for $|\gamma| < 2$. In [65] and subsequent work in its series, J. Miller and S. Sheffield prove that for the value $\gamma = \sqrt{8/3}$, the LQG coincides with the Brownian map, which is a random metric measure space arising as a universal scaling limit of random trees and random planar graphs (see [57, 59] and the references therein). More recently, [21, 41] establish the existence of the LQG metric for $\gamma \in (0, 2)$. We also note that the case where γ is complex valued is studied in [39, 71].

Geometric context

Despite the fact that the main attention of the probability community has focused so far on the two-dimensional case, (nonrandom)conformal geometry in dimensions $n > 2$ is a fascinating field of research. Earlier results by [3, 76, 88] completely solve the Yamabe problem [91] on compact manifolds: every compact Riemannian manifold is conformally equivalent to a manifold with constant scalar curvature. In the general case, despite ground-breaking results by [35] using the conformal Laplacian, a complete picture is still far from reach. On surfaces, the works [67, 68] initiate an approach to the problem based on Polyakov's variational formulation for the determinant of Δ_g [73, 74]: they show that constant curvature metrics have maximum determinant. In dimension 4, [11] derives an equivalent of Polyakov's formula for a conformal version of Δ_g^2, known as the Paneitz operator and [19] finds extremal metrics associated to some functionals of the conformal Laplacian and the Paneitz operator. Graham et al.[40] construct higher order equivalent of Paneitz operators, which is conformally invariant powers of Δ_g, based on [36], see also [49]. In particular, in dimension 4, remarkable spectral properties, sharp functional inequalities, and rigidity results have been derived in [17, 19, 48] and [18]. See also [24] for various such results in higher dimensions.

Higher dimensional random geometry

Conformally (quasi-)invariant extension for any of these random objects to higher dimensions was also discussed in [60] and [16]. Indeed, until we finished and circulated a first version of our paper, we were not aware of any of these contributions. The ansatz of B. Cerclé [16] is similar to ours, limited, however, to the sphere in $\mathbb{R}^{n+1}$ and relying on an *extrinsic* approach, based on stereographic projections of the Euclidean space, whereas ours is an intrinsic, Riemannian approach. In particular, our approach also applies to huge classes of manifolds with negative total Q-curvature, a necessary condition for finiteness of the partition function and for well-definedness of the (normalized) Polyakov–Liouville measure. The approach by T. Levy and Y. Oz [60] does not provide mathematical results or insights. It is more on a heuristic level, not taking care, however, of the necessary positivity of the respective GJMS operators, and not addressing any details of the necessary renormalization procedure. Our intrinsic Riemannian approach also has the advantage that it canonically provides approximations by discrete polyharmonic fields and associated Liouville measures [23]. Our construction of Liouville Brownian motion in higher dimensions and random GJMS operators is not anticipated so far, even not for the sphere or other particular cases.

Beyond that, the authors in [30] carry out the first major step toward an LQG metric on Euclidean space with arbitrary dimension. They prove tightness of the exponential metrics for log-correlated Gaussian fields on $\mathbb{R}^n$, generalizing the result in [21].

1 | CO-POLYHARMONIC OPERATORS ON EVEN-DIMENSIONAL MANIFOLDS

Throughout the sequel, without explicitly mentioning it, all manifolds under consideration are assumed to be smooth, connected, and closed (i.e., compact and without boundary).

1.1 | Riemannian manifolds and conformal classes

Given a closed Riemannian manifold (M, g), we denote its dimension by n, its volume measure by $\mathsf{vol} = \mathsf{vol}_g$, its scalar curvature by scal or by R, its Ricci curvature tensor by $\mathsf{Ric} = \{\mathsf{Ric}_{ij} : i, j = 0, \dots, n\}$, and its Laplace–Beltrami operator by $\Delta = \Delta_g$, the latter being a negative operator. The spectral gap (or in other words, the first nontrivial eigenvalue) of $-\Delta_g$ on (M, g) is denoted by $\lambda_1 > 0$.

For $u \in L^1(M, \mathsf{vol}_g)$, we set $\langle u \rangle_g := \frac{1}{\mathsf{vol}_g(M)} \int_M u \, d\mathsf{vol}_g$ and $\pi_g(u) := u - \langle u \rangle_g$. We use the short hand notation $L^2_g := L^2(M, \mathsf{vol}_g)$ and define the *grounded* L^2 *space* by

$$\mathring{L}^2_g := \{u \in L^2_g : \langle u \rangle_g = 0\}.$$

Definition 1.1.

(i) Two Riemannian metrics g and g' on a manifold M are *conformally equivalent* if there exists a (“weight”) function $\varphi \in C^\infty(M)$ such that $g' = e^{2\varphi} g$. The class of metrics that are conformally equivalent to a given metric g is denoted by $[g]$.

(ii) Two Riemannian manifolds (M, g) and (M', g') are *conformally equivalent* if there exists a C^∞-diffeomorphism $\Phi : M \to M'$ and a function $\varphi \in C^\infty(M)$ such that the pull back of g' is conformally equivalent to g with weight φ, that is

$$\Phi^* g' = e^{2\varphi} g \ .$$

In other words, if (M', g') is isometric to (M, g''), and g'' and g are conformally equivalent. The class of Riemannian manifolds that are conformally equivalent to a given Riemannian manifold (M, g) is denoted by $[(M, g)]$.

(iii) A family of operators A_g on a family of conformally equivalent Riemannian manifolds (M, g) is called *conformally quasi-invariant* if for every pair (M, g) and (M', g') of conformally equivalent manifolds and associated maps Φ and φ as in (ii) there exists a function f_φ on M such that

$$e^{f_\varphi} \cdot (A_{e^{2\varphi}g} u)\circ\Phi = A_g(u\circ\Phi) \,, \qquad u \in C^\infty(M) \,. \tag{11}$$

In conformal geometry, such an operator is usually called *conformally covariant*. However, in this paper, the notion *covariance* is already used for the key quantity for characterizing probabilistic dependencies.

The study of conformal mappings as in (ii) above is of particular interest in dimension 2 as powerful uniformization results are available. For instance, Riemann's mapping theorem [69] states that every nonempty simply connected open strict subset of $\mathbb{C}$ is conformally equivalent to the open unit disk. More generally, the uniformization theorem [72] asserts that every simply connected Riemann surface is conformally equivalent either to the sphere, the plane, or the disc (each of them equipped with its standard metric).

In contrast, the class of conformal mappings in higher dimensions is very limited. According to Liouville's theorem [58], conformal mappings of Euclidean domains in dimension $\geqslant 3$ can be expressed as a finite number of compositions of translations, homotheties, orthonormal transformations, and inversions.

Example 1.2. Let (M', g') be the complex plane and (M, g) be the 2-sphere without north pole n, regarded as a punctured Riemann sphere. Then they are conformally equivalent in the sense of Definition 1.1 (ii). The conformal map Φ is given by the stereographic projection (i.e., for all x on the sphere $\Phi(x)$ is the stereographic projection of the point x), and the weight φ is given by $\varphi(x) = -2\log(\sqrt{2}\sin(d_{\mathbb{S}^2}(\mathsf{n}, x)/2))$. This example, however, does not fit the setting of this work in two respects: (1) the manifold M' is noncompact and (2) the weight φ is nonsmooth on the completion of M (it has a singularity at the north pole).

1.2 | Co-polyharmonic operators

Henceforth, n denotes an *even* number and (M, g) is a *closed* Riemannian manifold of dimension n.

Our interest is primarily in the case $n \geqslant 4$. The case $n = 2$ is widely studied with celebrated, deep, and fascinating results. It serves here as a guideline. In this case, most of the following constructions and results are (essentially) well known.

The fundamental object for our subsequent considerations is the *co-polyharmonic operators* P_g, also called *conformally covariant powers of the Laplacian* or *Graham–Jenne–Mason–Sparling operators of maximal order* (i.e., of order $n/2$) as introduced in [40]. The *co*-polyharmonic operators are *com*panions of the polyharmonic operators $(-\Delta_g)^{n/2}$, coming with *cor*rection terms that make them *con*formally invariant. The construction of the co-polyharmonic operators P_g is quite involved. We outline this construction in Section 1.3. Before we get into that, let us first summarize the crucial properties of the operators P_g that is relevant for the sequel. We stress that, together with the sign convention $\Delta_g \leqslant 0$, our definition (3) implies that P_g always has nonnegative principal part.

Theorem 1.3. *For every closed manifold (M, g) of even dimension n,*

(i) *the co-polyharmonic operator P_g is a differential operator of order n,*
(ii) *the leading order is $(-\Delta_g)^{n/2}$, the zeroth-order vanishes,*
(iii) *the coefficients are C^∞ functions of the curvature tensor and its derivatives,*
(iv) *it is symmetric and extends to a self-adjoint operator, the Friedrichs extension, denoted by the same symbol, on $L^2(M, \mathsf{vol}_g)$ with domain $\mathcal{H}^n(M, \mathsf{vol}_g)$,*
(v) *it is conformally quasi-invariant: if $g' = e^{2\varphi} g$ for some $\varphi \in C^\infty(M)$, then*

$$\mathsf{P}_{g'} = e^{-n\varphi} \mathsf{P}_g. \tag{12}$$

More generally, assume that (M, g) and (M', g') are conformally equivalent with C^∞-diffeomorphism $\Phi : M \to M'$ and weight $\varphi \in C^\infty(M)$ such that $\Phi^* g' = e^{2\varphi} g$. Then,

(vi) *for all $u \in C^\infty(M')$:*

$$\big(\mathsf{P}_{M',g'} u\big)\circ\Phi = e^{-n\varphi} \mathsf{P}_{M,g}(u\circ\Phi). \tag{13}$$

Proof. Most properties are due to [40], and restated in [49]; self-adjointness is proven in [49, Corollary, p. 91]. □

Remark 1.4.

(a) Some authors work directly with a Laplacian defined as a nonnegative operator (e.g., [49]). Other authors work with the usual Laplacian and consider the operator P_g with leading term $\Delta_g^{n/2}$ (e.g., [13, 42, 53]); this would correspond to $(-1)^{n/2}\mathsf{P}_g$ in our convention.
(b) In general, no closed expressions exist for the operators P_g. However, recursive formulas for the expression of P_g are known and a priori allow to explicitly compute P_g for any even n, [53]. As the dimension increases, these formulas become more and more involved, as the complexity of lower-order terms grows exponentially with n.

Proposition 1.5. *The most prominent cases are:*

(i) *If $n = 2$, then $\mathsf{P}_g = -\Delta_g$.*
(ii) *If $n = 4$, then $\mathsf{P}_g = \Delta_g^2 + \mathsf{div}\left(2\mathsf{Ric}_g - \frac{2}{3}\mathsf{scal}_g\right)\nabla$ is the celebrated* Paneitz operator, *see [70]. Here the curvature term $2\mathsf{Ric}_g - \frac{2}{3}\mathsf{scal}_g$ should be viewed as an endomorphism of the tangent bundle, acting on the gradient of a function. In coordinates:*

$$\mathsf{P}_g u = \sum_{i,j} \nabla_i \left[\nabla^i \nabla^j + 2\mathsf{Ric}_g^{ij} - \frac{2}{3}\mathsf{scal}_g \cdot g^{ij}\right] \nabla_j u, \qquad \forall u \in C^\infty(M).$$

(iii) *If (M, g) is an* Einstein manifold *with $\mathsf{Ric}_g = kg$ (for some $k \in \mathbb{R}$) and even dimension n, then*

$$\mathsf{P}_g = \prod_{j=1}^{n/2} \left[-\Delta_g + \frac{k}{n-1}\nu_j^{(n)}\right] \tag{14}$$

with $\nu_j^{(n)} := \frac{n}{2}(\frac{n}{2} - 1) - j(j-1) = (\frac{n-1}{2})^2 - (\frac{2j-1}{2})^2$ for $j = 1, \dots, n/2$.

Proof. For (i) and (ii), see [40] or [15, p. 122]; for (iii), see [42, Thm. 1.2]. All formulas above appear in these references up to a factor $(-1)^{n/2}$, due to the sign convention in the definition of P_g. □

Example 1.6. If (M, g) is flat, then $\mathsf{P}_g = (-\Delta_g)^{n/2}$ is the positive *poly-Laplacian*.

Example 1.7. If (M, g) is the round sphere $\mathbb{S}^n$, then $\mathsf{P}_g = \prod_{j=1}^{n/2}[-\Delta_g + \nu_j^{(n)}]$ with $\nu_j^{(n)}$ as above (this formula already appears in [13]). In particular, $\mathsf{P}_g = \Delta_g^2 - 2\Delta_g$ in the case $n = 4$, and $\mathsf{P}_g = -\Delta_g^3 + 10\Delta_g^2 - 24\Delta_g$ in the case $n = 6$.

Conformally invariant operators with leading term a power of the Laplacian Δ_g have been a focus in mathematics and physics for decades. For instance, Dirac [26] constructs a conformally invariant wave operator on a four-dimensional surface in the five-dimensional projective plane in order to show that Maxwell equations are conformally invariant in a curved space time (Lorentzian manifolds). In the case of Riemannian manifolds, the Yamabe operator

$$\Delta_g - \frac{n-2}{4(n-1)}\mathsf{scal}_g \ ,$$

encodes the behavior of the Ricci curvature under conformal change and has proved of uttermost importance in the resolution of the Yamabe problem on compact Riemannian manifolds [3, 76, 88, 91]. Paneitz [70] constructs a conformally invariant operator with leading term Δ_g^2, and sixth-order analogs are constructed in [12, 90].

1.3 | Construction of the co-polyharmonic operators

Now let us outline the construction of the co-polyharmonic operators, introduced by C.R. Graham, R. Jenne, L.J. Mason, and G.A.J. Sparling [40]. They base their original construction on the *ambient metric*, introduced by C. Fefferman and C.R. Graham [36], a Lorentzian metric on a suitable manifold of dimension $n+2$. In an alternative approach, proposed by C.R. Graham and M. Zworski [49], the manifold (M, g) is regarded as the *boundary at infinity* of an asymptotically hyperbolic Einstein manifold (N, h) of dimension $n+1$. Our presentation follows [49], focusing on manifolds of even dimension n and on operators with maximal degree $k = n/2$.

1.3.1 | The Poincaré metric associated to $(M, [g])$

Consider a closed Riemannian manifold (M, g) with conformal class $[g]$ and even dimension n. Choose an $n+1$-dimensional Riemannian manifold (N, h) with boundary such that $\partial N = M$, for instance, $N = [0, \infty) \times M$. A Riemannian metric h on N is called *conformally compact metric with conformal infinity* $[g]$ if

$$h = \frac{\overline{h}}{x^2}, \qquad \overline{h}|_{T\partial N} \in [g], \tag{15}$$

where $\overline{h}$ is a smooth metric on N, and $x : N \to \mathbb{R}_+$ is a smooth function such that $\{x = 0\} = \partial N$ and $dx|_{\partial N} \neq 0$. We say that the metric h is *asymptotically even* if it is given as

$$h = \frac{1}{x^2}\left[dx^2 + \sum_{i,j=1}^{n} \overline{h}_{ij}(x,\xi)d\xi^i d\xi^j\right],$$

where x is as above, $(\xi^1, \dots, \xi^n)$ forms a coordinate system on M, and $\overline{h}_{ij}$ for $1 \leqslant i, j \leqslant n$ is an even function of x.

Definition 1.8. A *Poincaré metric associated to* $[g]$ is a conformally compact metric h with conformal infinity $[g]$ that is asymptotically even and satisfies

$$\mathsf{Ric}_g + ng = \mathcal{O}(x^{n-2}), \qquad \mathsf{tr}_g\left(\mathsf{Ric}_g + ng\right) = \mathcal{O}(x^{n+2}). \tag{16}$$

Lemma 1.9 [36, Theorem 2.3]. *For every closed $(M, [g])$ of even dimension n, there exists a Poincaré metric h. It is uniquely determined up to addition of terms vanishing to order $n-2$ and up to a diffeomorphism fixing M.*

The prime example for this construction is provided by the Poincaré model of hyperbolic space: the n-dimensional round sphere $M = \mathbb{S}^n$ is the boundary of the unit ball $N = B_1(0) \subset \mathbb{R}^{n+1}$ equipped with the hyperbolic metric $dh(r, \xi) = \frac{4}{(1-r^2)^2}[dr^2 + dg(\xi)]$.

1.3.2 | The generalized Poisson operator on (N, h)

Consider a closed $(M, [g])$ of even dimension n and let h be the Poincaré metric associated to it on a suitable N with $\partial N = M$. Extending the traditional Landau notation, for a function v on N, we say that $v = \mathcal{O}(x^\infty)$ if $v = \mathcal{O}(x^a)$ as $x \to 0$ for every $a \in \mathbb{N}$.

Lemma 1.10 [49, Props. 4.2, 4.3].

(i) *For every $f \in C^\infty(M)$, there exists a solution to*

$$\Delta_h u = \mathcal{O}(x^\infty) \tag{17}$$

of the form

$$u = F + G\, x^n \log x\ , \qquad F, G \in C^\infty(N)\ , \qquad F|_M = f\ . \tag{18}$$

Here, F is uniquely determined mod $\mathcal{O}(x^n)$ *and G is uniquely determined* mod $\mathcal{O}(x^\infty)$.

(ii) *Put* $\sigma_n := (-1)^{n/2}\, 2^n (n/2)!\, (n/2-1)!$ *and*

$$\mathsf{P}_g f := -2\sigma_n\, G|_M. \tag{19}$$

Then P_g is a differential operator on M with principal part $(-\Delta_g)^{n/2}$. It only depends on g and defines a conformally invariant operator that agrees with the operator constructed in [40].

(No sign adjustment is required in comparison with [49] since the convention there is that the Laplace–Beltrami operator is nonnegative.)

Remark 1.11. Given $(M, [g])$ and (N, h) as above, the co-polyharmonic operator P_g can alternatively be defined as residue at $s = n$ of the meromorphic family of scattering matrix operators

$S(s)$, $s \in \mathbb{C}$, on (N, h),

$$\mathsf{P}_g = -2\sigma_n \operatorname{Res}_{s=n} S(s) \tag{20}$$

with σ_n as above [49, Thm. 1].

1.4 | Branson's Q-curvature

Co-polyharmonic operators are closely related to *Branson's Q-curvature* in even dimension, another important notion in conformal geometry.

The notion of Q-curvature was introduced on arbitrary even-dimensional manifolds by T. Branson [12, p. 11]. Its construction and properties have since been studied by many authors. In dimension 4, explicit computations for the Q-curvature are due to T. Branson and B. Ørsted [11]. Its properties are very much akin to those of scalar curvature in two dimensions. C. Fefferman and K. Hirachi [37] presented an approach based on the ambient Lorentzian metric of [36]. The definition of Q-curvature may differ in the literature up to a sign or a factor 2. Following [49], with the notation of Remark 1.11, we have $Q_g = -2\sigma_n S(n)\mathbf{1}$.

The crucial property of Q-curvature is its behavior under conformal transformations.

Proposition 1.12 [13, Corollary 1.4]. *If $g' = e^{2\varphi} g$, then*

$$e^{n\varphi} Q_{g'} = Q_g + \mathsf{P}_g \varphi. \tag{21}$$

Our sign convention for Q_g comes from our sign convention for P_g together with the validity of Equation (21).

Corollary 1.13. *The* total Q-curvature$Q(M, g) := \int_M Q_g \,\mathrm{dvol}_g$ *is a conformal invariant.*

Proof. By the previous proposition,

$$\begin{aligned} Q(M, g') = \int_M Q_{g'} e^{n\varphi} \mathrm{dvol}_g &= Q(M, g) + \int_M \mathsf{P}_g \varphi \,\mathrm{dvol}_g \\ &= Q(M, g) + \int_M \varphi \,\mathsf{P}_g \mathbf{1} \,\mathrm{dvol}_g = Q(M, g) \end{aligned}$$

due to the self-adjointness of P_g and the fact that it annihilates constants. □

Again, explicit formulas are only known in low dimensions or for Einstein manifolds.

Example 1.14. Important cases are

(i) If $n = 2$, then $Q_g = \frac{1}{2}\mathsf{R}_g = \frac{1}{2}\mathsf{scal}_g$ is half of the *scalar curvature*, see, for example, [15, Eqn. 3.1, up to a factor $-1 = (-1)^{n/2}$].

(ii) If $n = 4$, then $Q_g = -\frac{1}{6}\Delta_g \mathsf{scal}_g - \frac{1}{2}|\mathsf{Ric}_g|^2 + \frac{1}{6}\mathsf{scal}_g^2$ with $|\mathsf{Ric}_g|^2 = \sum_{i,j} \mathsf{Ric}^{ij}\mathsf{Ric}_{ij}$, see [11].

(iii) If (M, g) is an Einstein manifold with $\mathrm{Ric}_g = k\, g$ and even dimension, then [42, Thm. 1.1, up to a factor $(-1)^{n/2}$]

$$Q_g = (n-1)! \left(\frac{k}{n-1}\right)^{n/2} . \tag{22}$$

In particular, for the round sphere, $Q_g = (n-1)!$. For instance, if $n = 4$, then $Q = 6$.

Recall that a Riemannian manifold is called *conformally flat* if it is conformally equivalent to a flat manifold.

Proposition 1.15. *Let $\chi(M)$ denote the Euler characteristic of (M, g).*

(i) *In the case $n = 2$,*

$$Q(M, g) = 2\pi\, \chi(M) .$$

(ii) *In the case $n = 4$,*

$$Q(M, g) = 8\pi^2 \chi(M) - \frac{1}{4} \int_M |W|^2 \mathrm{dvol}_g ,$$

where W is the Weyl tensor, and $|W|^2 = \sum_{a,b,c,d} W^{abcd} W_{abcd}$. In particular,

$$Q(M, g) = 8\pi^2 \chi(M) \iff (M, g) \text{ is conformally flat.}$$

(iii) *For any even n, if (M, g) is conformally flat, then with $c_n = \frac{1}{2}(2n-1)!(4\pi)^{n/2}$,*

$$Q(M, g) = c_n\, \chi(M) .$$

Proof. The two-dimensional claim follows from Example 1.14 and the Gauss–Bonnet theorem. See [11, p. 673] for the case of dimension four, and [49, p. 3] for the case of conformally flat manifolds in even dimension. Alternatively, see [15, pp. 122f., up to a factor $(-1)^{n/2}$]. □

Remark 1.16. Recall that for two-dimensional oriented Riemannian manifolds, $\chi(M) = 2 - 2\mathfrak{g}$ where $\mathfrak{g}$ denotes the genus of M. Furthermore, for the sphere in even dimension, $\chi(\mathbb{S}^n) = 2$.

1.4.1 | Some rigidity and equilibration results in $n = 4$

In dimension 4, the conformal invariant integral of the Q-curvature leads to remarkable rigidity and equilibration results, resembling famous analogous results in dimension 2. To formulate them, let us introduce another important conformal invariant, the Yamabe constant

$$Y(M, g) := \inf_{h \in [g]} \frac{\int_M \mathrm{scal}_h \mathrm{dvol}_h}{\sqrt{\mathrm{vol}_h(M)}} .$$

Proposition 1.17 [19, 48]. *Assume $n = 4$.*

(i) *There exist closed hyperbolic manifolds with $Y(M, g) < 0$ and $Q(M, g) > 16\pi^2$.*
(ii) *If $Y(M, g) \geqslant 0$, then $Q(M, g) \leqslant 16\pi^2$ with equality if and only if $M = \mathbb{S}^4$.*
(iii) *If $Y(M, g) \geqslant 0$ and $Q(M, g) \geqslant 0$, then $\mathsf{P}_g \geqslant 0$ and $\mathsf{P}_g u = 0 \iff u$ is constant.*
(iv) *If $Q(M, g) \leqslant 16\pi^2$ and $\mathsf{P}_g \geqslant 0$ with $\mathsf{P}_g u = 0 \iff u$ is constant, then there exists a conformal metric g' with constant Q-curvature.*

Proposition 1.18 [63, Theorem 4.1]. *For any $g_0 = e^{2\varphi_0} g$ on $M = \mathbb{S}^4$, the Q-curvature flow*

$$\frac{\partial}{\partial t} g_t = -2(Q_{g_t} - \bar{Q}_{g_t}) g_t$$

(with $\bar{Q}_{g_t} := \langle Q_{g_t} \rangle_{g_t}$ the mean value of Q_{g_t} on $(\mathbb{S}^4, g_t)$) converges exponentially fast to a metric $g_\infty = e^{2\varphi_\infty} g$ of constant Q-curvature 6 in the sense that $\|\varphi_t - \varphi_\infty\|_{H^4} \leqslant C\, e^{-\delta t}$ for some constants C and $\delta > 0$.

2 | ADMISSIBILITY, SOBOLEV SPACES, AND KERNEL ESTIMATES

2.1 | Admissible manifolds

Definition 2.1. We say that a Riemannian manifold (M, g) is *admissible* if it is closed and of even dimension, and if the co-polyharmonic operator P_g is positive definite on $\mathring{L}^2(M, \mathrm{vol}_g)$.

As an immediate consequence of Theorem 1.3 (v), we obtain the following.

Corollary 2.2. *Admissibility of a Riemannian manifold (M, g) is a conformal invariance, or in other words, it is a property of the conformal class $(M, [g])$.*

More generally, admissibility of (M, g) implies admissibility of any (N, h) conformally equivalent to (M, g) in the sense of Definition 1.1 (ii).

Example 2.3. Every closed two-dimensional manifold is admissible.

Having at hand the explicit representation formula for the co-polyharmonic operators on Einstein manifolds from Lemma 1.5, we easily conclude the following.

Proposition 2.4. *Every closed even-dimensional Einstein manifold with nonnegative Ricci curvature is admissible.*

More generally, we obtain:

Proposition 2.5. *A closed Einstein manifold of even dimension n and of Ricci curvature $-(n-1)\kappa$ is admissible if and only if $\lambda_1 > \frac{n(n-2)}{4}\kappa$.*

Proof. Since M has constant Ricci curvature $-(n-1)\kappa$, according to Proposition 1.5, $\mathsf{P}_g = \prod_{j=1}^{n/2}[-\Delta_g - \kappa\nu_j^{(n)}]$ with ν_j ranging between 0 and $\frac{n}{2}(\frac{n}{2}-1)$. Thus, $\mathsf{P}_g > 0$ on $\mathring{L}^2(M, \mathrm{vol}_g)$ if and only if $\lambda_1 > \frac{n}{2}(\frac{n}{2}-1)\kappa$. □

Remark 2.6.

(a) The number $\frac{n(n-2)}{4}$ is strictly smaller than $\frac{(n-1)^2}{4}$ that plays a prominent role as threshold for the spectral gap of hyperbolic manifolds (and which is also the spectral bound for the simply connected hyperbolic space). Many results in hyperbolic geometry deal with the question whether λ_1 is close to $\frac{(n-1)^2}{4}$.

(b) The *Elstrodt–Patterson–Sullivan theorem* [84, Thm. (2.17)] provides a lower bound for λ_1 for a hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ in terms of the *critical exponent* $\delta(\Gamma)$ of the Kleinian group Γ acting on the simply connected hyperbolic space $\mathbb{H}^n$ of dimension n and curvature -1. More precisely,

$$\lambda_1 > \frac{n(n-2)}{4} \quad \text{if (and only if) } \delta(\Gamma) < \frac{n}{2}\,, \tag{23}$$

and, moreover, $\lambda_1 = \frac{(n-1)^2}{4}$ if (and only if) even $\delta(\Gamma) \leqslant \frac{n-1}{2}$. Here, $\delta(\Gamma)$ denotes the infimal value for which the Poincaré series for Γ converges, that is,

$$\delta(\Gamma) := \inf\left\{ s \in \mathbb{R} : \sum_{\gamma\in\Gamma} \exp\left(-s\, d(x, \gamma y)\right) < \infty \right\},$$

the latter being independent of the choice of $x, y \in M$.

(c) Similar estimates for λ_1 exist in terms of the Hausdorff dimension D of the *limit set of* Γ, provided that Γ is *geometrically finite without cusps*, see [84, Thm. (2.21)]. More precisely,

$$\lambda_1 > \frac{n(n-2)}{4} \quad \text{if (and only if) } D < \frac{n}{2}\,, \tag{24}$$

and, moreover, $\lambda_1 = \frac{(n-1)^2}{4}$ if (and only if) even $D \leqslant \frac{n-1}{2}$.

Proposition 2.7. *For every even dimension $n \geqslant 4$, there exist closed Einstein manifolds that are not admissible. They can be constructed, for instance, as $M = M_1 \times M_2$ where M_1 denotes any closed manifold of dimension $n-2$ and of constant curvature $-\frac{1}{n-3}$, and where M_2 denotes any closed hyperbolic Riemannian surface with $\lambda_1(M_2) \leqslant 2/3$.*

Remark 2.8. According to [14, Satz 1], for every $\varepsilon > 0$, there exist closed hyperbolic Riemannian surfaces with genus 2 and $\lambda_1 < \varepsilon$.

Proof of Proposition 2.7. By construction, M is an Einstein manifold with constant Ricci curvature $-g$. Thus, by Proposition 2.5, M is admissible if and only if $\lambda_1(M) > \frac{n(n-2)}{4(n-1)} \geqslant \frac{2}{3}$. On the other hand, by construction, $\lambda_1(M) \leqslant \lambda_1(M_2) \leqslant \frac{2}{3}$. □

2.2 | Estimates for heat kernels and resolvent kernels

Our main result in the section, Theorem 2.19, provides a sharp asymptotic estimate for the integral kernel of P_g^{-1} on $\mathring{L}^2(M, \mathsf{vol}_g)$. Deriving this requires precise estimates on the integral kernel of the operators $(\alpha - \Delta)^{-n/2}$ for $\alpha > -\lambda_1$. These estimates, in turn, depend on sharp heat kernel estimates, the upper one of which is new.

For a Riemannian manifold (M, g), denote by sec its sectional curvature, by inj its injectivity radius, and by p_t its heat kernel, that is, the integral kernel of the heat semigroup $P_t := e^{t\Delta}$.

Proposition 2.9. *Let (M, g) be a closed n-dimensional manifold.*

(i) *Assume that $\mathsf{Ric}_g \geqslant -(n-1)a^2\, g$ and set $\lambda_* := \frac{(n-1)^2}{4}a^2$ if $n \neq 2$ and $\lambda_* = \frac{1}{6}a^2$ if $n = 2$. Then, for all $t > 0$ and all $x, y \in M$,*

$$p_t(x,y) \geqslant \frac{1}{(4\pi t)^{n/2}} \left(\frac{a\, d(x,y)}{\sinh(a\, d(x,y))} \right)^{\frac{n-1}{2}} e^{-\frac{d^2(x,y)}{4t}} e^{-\lambda_* t}. \tag{25}$$

(ii) *Let a ball $B = B_R(x) \subset M$ be given, assume that $\sec \leqslant b^2$ on B and that $\mathsf{inj}_x \geqslant R$, and let p_t^0 denote the heat kernel on B with Dirichlet boundary conditions. Moreover,*
- *in the case $n \neq 2$, assume that $R \leqslant \frac{\pi}{b}$, and set $\lambda^* := \frac{n(n-1)}{6} b^2$,*
- *in the case $n = 2$, assume that $R \leqslant \frac{\pi}{2b}$, and set $\lambda^* := \frac{1}{2} b^2$.*

Then, for all $t > 0$ and all $y \in B$,

$$p_t^0(x,y) \leqslant \frac{1}{(4\pi t)^{n/2}} \left(\frac{b\, d(x,y)}{\sin(b\, d(x,y))} \right)^{\frac{n-1}{2}} e^{-\frac{d^2(x,y)}{4t}} e^{+\lambda^* t}. \tag{26}$$

Proof.

(i) follows from [83, Cor. 4.2 and Rmk. 4.4(a)] (we work with the geometric heat semigroup $e^{t\Delta}$ rather than with the probabilistic semigroup $e^{t\frac{\Delta}{2}}$ as in [83]).

(ii) Let $\bar{M} = \mathbb{S}^{b,n}$ denote the round sphere of dimension n and radius $1/b$ (which has constant curvature b^2), fix a point $\bar{x} \in \bar{M}$, and let $\bar{B}$ denote the ball around $\bar{x}$ of radius R in $\bar{M}$. Denote by $\bar{p}_t^0$ the heat kernel on $\bar{B}$ with Dirichlet boundary conditions. By rotational invariance,

$$\bar{p}_t^0(\bar{x}, \bar{y}) = \bar{p}_t^0\big(\bar{d}(\bar{x}, \bar{y})\big)$$

for some function $r \mapsto \bar{p}_t^0(r)$. According to the celebrated heat kernel comparison theorem of Debiard–Gaveau–Mazet [22],

$$p_t^0(x,y) \leqslant \bar{p}_t^0(d(x,y)) \tag{27}$$

for all $t > 0$ and all $y \in B$.

We treat the case $n \neq 2$ first. Following the strategy for deriving the lower bound (25) in [83], define

$$\hat{p}_t^0(r) := \frac{1}{(4\pi t)^{n/2}} \left(\frac{b\, r}{\sin(b\, r)} \right)^{\frac{n-1}{2}} e^{-\frac{r^2}{4t}} e^{\lambda^* t} = g_t(r) \left(\frac{br}{\sin(br)} \right)^{\frac{n-1}{2}} e^{\lambda^* t}.$$

where λ^* as defined above and $g_t(r) = (4\pi t)^{-n/2} e^{-r^2/4t}$ is the Gaussian kernel. We show that the function $(t, \bar{y}) \mapsto H(t, \bar{y}) := \hat{p}^0_t(\bar{d}(\bar{x}, \bar{y}))$ is space-time superharmonic on $(0, \infty) \times \bar{B}$. Indeed, a direct computation yields:

$$\partial_t \log \hat{p}^0 = \partial_t \log g + \lambda^*;$$

$$\partial_r \log \hat{p}^0 = \partial_r \log g + \frac{n-1}{2}\left(\frac{1}{r} - b\frac{\cos br}{\sin br}\right);$$

$$\partial^2_{rr} \log \hat{p}^0 = \partial^2_{rr} \log g + \frac{n-1}{2}\left(\frac{b^2}{\sin^2 br} - \frac{1}{r^2}\right).$$

Now using that H is a radial function and the chain rule we find that

$$\frac{1}{H}(\partial_t - \bar{\Delta})H = \partial_t \log \hat{p}^0 - (n-1)b\,\frac{\cos br}{\sin br}\partial_r \log \hat{p}^0 - \partial^2_{rr} \log \hat{p}^0 - \left(\partial_r \log \hat{p}^0\right)^2 , \tag{28}$$

where the left-hand side is evaluated at $(t, \bar{y})$ and the right-hand side at $(t, \bar{d}(\bar{x}, \bar{y}))$. We easily verify that g satisfies:

$$\partial_t \log g - \partial^2_{rr} \log g - (\partial_r \log g)^2 - \frac{n-1}{r}\partial_r \log g = 0 .$$

We thus see that in (28), all the appearances of $\log g$ cancel out, and we get:

$$\begin{aligned}
\frac{1}{H}(\partial_t - \bar{\Delta})H &= \lambda^* - \frac{n-1}{2}\left(\frac{b^2}{\sin^2 br} - \frac{1}{r^2}\right) \\
&\quad - \frac{(n-1)^2}{2} b\frac{\cos br}{\sin br}\left(\frac{1}{r} - b\frac{\cos br}{\sin br}\right) - \frac{(n-1)^2}{4}\left(\frac{1}{r} - b\frac{\cos br}{\sin br}\right)^2 \\
&= \lambda^* - \frac{(n-1)(n-3)}{2}\frac{1}{r^2} + b^2\frac{\cos^2 br}{\sin^2 br}\frac{(n-1)^2}{4} - \frac{n-1}{2}b^2\frac{1}{\sin^2 br} \\
&= \lambda^* + \frac{(n-1)(n-3)}{4}\left[\frac{b^2}{\sin^2 br} - \frac{1}{r^2}\right] - b^2\frac{(n-1)^2}{4} \\
&\geqslant \lambda^* + \frac{(n-1)(n-3)}{4}\frac{b^2}{3} - b^2\frac{(n-1)^2}{4} = 0 .
\end{aligned}$$

On the other hand, $\bar{p}$ is harmonic and by a comparison principle for solution of parabolic equations, we thus have that $p^0 \leqslant H$. In order to properly justify the comparison principle, instead of working with p^0 and H that have singular initial condition, we work instead with $p^0_{R'}$ the solution to the heat equation with initial condition $\mathbf{1}_{B_{R'}(y)}$ and $H_{R'}$ that has the same expression as H except that we choose

$$g_{R'}(t, r) = \int_{B^{\mathbb{R}^n}_{R'}(r)} (4\pi t)^{-\frac{n}{2}} e^{-|y|^2 4t}\, dy ,$$

instead of g. The same computation yields that $H_{R'}$ is a supersolution to the heat equation. Then, we argue as in [83].

Now, for the case $n = 2$, defining $\hat{p}^0$ and H as above, we still find that

$$\frac{1}{H}(\partial_t - \bar{\Delta})H = \lambda^* - \frac{1}{4}\left[\frac{b^2}{\sin^2 br} - \frac{1}{r^2} + b^2\right]$$
$$\geqslant \lambda^* - b^2\left[\frac{1}{2} - \frac{1}{\pi^2}\right] \geqslant 0 \; ,$$

where we used that $r < \frac{\pi}{2b}$. The rest of the proof is similar. □

Before stating our main estimates, let us introduce some notation and provide some auxiliary results.

Lemma 2.10.

(i) *For every $\alpha > 0$ and $s > 0$, the* resolvent operator $\mathsf{G}_{s,\alpha} := (\alpha - \Delta)^{-s}$ *on* $L^2 = L^2(M, \mathrm{vol}_g)$ *is an integral operator with kernel given by*

$$G_{s,\alpha}(x,y) := \frac{1}{\Gamma(s)} \int_0^\infty e^{-\alpha t}\, t^{s-1}\, p_t(x,y)\, dt.$$

Since $\langle P_t u\rangle_g = \langle u\rangle_g$ for all u, the heat operator $P_t = e^{t\Delta}$ also acts on the grounded L^2-space $\mathring{L}^2 = \{u \in L^2(M, \mathrm{vol}_g) : \langle u\rangle_g = 0\}$*, and so do the resolvent operators* $\mathsf{G}_{s,\alpha}$.

(ii) *Restricted to $\mathring{L}^2$, the* resolvent operator

$$\mathring{\mathsf{G}}_{s,\alpha} = (\alpha - \Delta)^{-s}\big|_{\mathring{L}^2}$$

is a compact, symmetric operator for every $\alpha > -\lambda_1$ and $s > 0$. It admits a symmetric integral kernel

$$\mathring{G}_{s,\alpha}(x,y) := \frac{1}{\Gamma(s)} \int_0^\infty e^{-\alpha t}\, t^{s-1}\, \mathring{p}_t(x,y)\, dt,$$

defined in terms of the grounded heat kernel $\mathring{p}_t(x,y) := p_t(x,y) - \mathrm{vol}_g(M)^{-1}$.

(iii) *By compactness of M, the operator $-\Delta$ has discrete spectrum $(\lambda_j)_{j\in\mathbb{N}_0}$, counted with multiplicity, and the corresponding eigenfunctions $(\chi_j)_{j\in\mathbb{N}_0}$ form an orthonormal basis for L^2_g. In terms of these spectral data, the grounded resolvent kernel is the symmetric kernel*

$$\mathring{G}_{s,\alpha}(x,y) = \sum_{j=1}^{\infty} \frac{\chi_j(x)\,\chi_j(y)}{(\alpha + \lambda_j)^s}. \tag{29}$$

(iv) *For every $s > n/2$ and $\alpha > 0$, there exists C such that for all $x, y \in M$,*

$$\left|\mathring{G}_{s,\alpha}(x,y)\right| \leqslant C \; , \tag{30}$$

and for every $s < n/2$ and $\alpha > 0$, there exists C such that for all $x, y \in M$,

$$G_{s,\alpha}(x,y) \leqslant \frac{C}{d(x,y)^{n-2s}} \; . \tag{31}$$

Proof. All of (i)–(iii) but (29) are proven by Strichartz [82, §4]. Regarding (29), by the spectral theorem, for all $u \in L^2$ and $\alpha > 0$ (or $u \in \mathring{L}^2$ and $\alpha > -\lambda_1$),

$$(\alpha - \Delta)^s u = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\alpha t}\, t^{s-1}\, P_t u\, dt = \sum_{j=0}^{\infty} \frac{\langle u \,|\, \chi_j \rangle_{L^2}}{(\alpha + \lambda_j)^s}\, \chi_j$$

with convergence of integral and sum in L^2 (or in $\mathring{L}^2$, resp.). Thus, (29) readily follows.

(iv) Estimate (30) is a consequence of [29, Cor. 6.2]. In order to show (31) fix $\varepsilon \ll 1$, $x, y \in M$ with $0 < r := d_g(x, y) < \varepsilon$, and set

$$G_{s,\alpha}(x, y) = \underbrace{\frac{1}{\Gamma(s)} \int_0^\varepsilon e^{-\alpha t} t^{s-1} p_t(x, y)\, \mathrm{d}t}_{I_1} + \underbrace{\frac{1}{\Gamma(s)} \int_\varepsilon^\infty e^{-\alpha t} t^{s-1} p_t(x, y)\, \mathrm{d}t}_{I_2} .$$

As a consequence of the upper heat kernel estimate [29, Eqn. (2.4)], there exists a constant $C = C(g, s, \alpha, \varepsilon) > 0$ independent of x, y, and such that $I_2 \leqslant C$ and

$$I_1 \leqslant C \int_0^\varepsilon e^{-\frac{r^2}{t}}\, t^{s-n/2-1}\, \mathrm{d}t .$$

Combining these estimates together,

$$G_{s,\alpha}(x, y) \leqslant C \left(1 + \int_0^\varepsilon e^{-\frac{r^2}{t}}\, t^{s-n/2-1}\, \mathrm{d}t \right) ,$$

and the assertion now follows from the known asymptotic expansion of the exponential integral function

$$E_{s-n/2+1}\left(\tfrac{r^2}{\varepsilon}\right) := \int_0^\varepsilon e^{-\frac{r^2}{t}}\, t^{s-n/2-1}\, \mathrm{d}t \asymp \Gamma(n/2 - s)\, r^{2s-n} \quad \text{as} \quad r \to 0 . \qquad \square$$

Remark 2.11. For all $s > 0$, the operators $\mathsf{G}_{s,\alpha}$ and $\mathring{\mathsf{G}}_{s,\alpha}$ are powers of $\mathsf{G}_\alpha := \mathsf{G}_{1,\alpha}$ and $\mathring{\mathsf{G}}_\alpha := \mathring{\mathsf{G}}_{1,\alpha}$, that is,

$$\mathsf{G}_{s,\alpha} = (\mathsf{G}_\alpha)^s, \qquad \mathring{\mathsf{G}}_{s,\alpha} = (\mathring{\mathsf{G}}_\alpha)^s.$$

Example 2.12. Let (M, g) be the two-dimensional round sphere $\mathbb{S}^2$. Then, according to [29, Thm. 6.12],

$$\mathring{G}_{1,0}(x, y) = -\frac{1}{4\pi} \left(1 + 2 \log \sin \frac{d(x, y)}{2} \right) .$$

Proposition 2.13. *Let (M, g) be a compact n-dimensional manifold and $\alpha > -\lambda_1$. Then, for all x and $y \in M$:*

$$\left| G_{n/2,\alpha}(x, y) - a_n \log \frac{1}{d(x, y)} \right| \leqslant C_0;$$

$$\left| \mathring{G}_{n/2,\alpha}(x, y) - a_n \log \frac{1}{d(x, y)} \right| \leqslant C_0;$$

for some $C_0 = C_0(g, \alpha) > 0$ *and*

$$a_n := \frac{2}{\Gamma(n/2)\,(4\pi)^{n/2}} . \tag{32}$$

Proof. For convenience, we split the proof.

Lower estimate for the ungrounded kernel
Take λ_* as in Proposition 2.9 (i) and $\alpha > \lambda_*$. For the nongrounded resolvent kernel, the lower heat kernel estimate (25) yields, with x and $y \in M$, and $r = d(x, y)$,

$$\begin{aligned} G_{n/2,\alpha}(x,y) &= \frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-\alpha t} p_t(x,y)\, t^{n/2-1} dt \\ &\geqslant \frac{1}{\Gamma(\frac{n}{2})\,(4\pi)^{n/2}} \left(\frac{ar}{\sinh(ar)}\right)^{\frac{n-1}{2}} \int_0^\infty e^{-(\alpha+\lambda_*)t}\, e^{-\frac{r^2}{4t}}\, \frac{dt}{t}. \end{aligned}$$

By [29, Eqn. (6.8)], for every $\beta > 0$:

$$\int_0^\infty e^{-\beta t}\, e^{-\frac{r^2}{4t}}\, \frac{dt}{t} = 4\pi \cdot G^{\mathbb{R}^2}_{1,\beta}(r) \geqslant 2 \log\frac{1}{r} - C_\beta .$$

Combining the two previous estimates yields

$$G_{n/2,\alpha}(x,y) - a_n \log\frac{1}{d(x,y)} > -C_{\alpha+\lambda^*} , \qquad x, y \in M .$$

Upper estimate for the ungrounded kernel with Dirichlet boundary conditions
Consider the case $\alpha > \lambda^*$, with λ^* as in Lemma 2.9 (ii). We estimate the contribution of p^0_t as before, with x and $y \in M$, and $r = d(x, y)$:

$$\begin{aligned} G^0_{n/2,\alpha}(x,y) &:= \frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-\alpha t}\, p^0_t(x,y)\, t^{n/2-1} dt \\ &\leqslant \frac{1}{\Gamma(\frac{n}{2})\,(4\pi)^{n/2}} \left(\frac{ar}{\sin(br)}\right)^{\frac{n-1}{2}} \int_0^\infty e^{(-\alpha+\lambda^*)t}\, e^{-\frac{r^2}{4t}}\, \frac{dt}{t} , \end{aligned}$$

and we can use the fact that, by [29], ibid.:

$$\int_0^\infty e^{-\beta t}\, e^{-\frac{r^2}{4t}}\, \frac{dt}{t} = 4\pi \cdot G^{\mathbb{R}^2}_{1,\beta}(r) \leqslant 2 \log\frac{1}{r} + C_\beta , \qquad \beta > 0 .$$

The two estimates yield

$$G^0_{n/2,\alpha}(x,y) - a_n \log\frac{1}{d(x,y)} < C_{\alpha-\lambda_*} , \qquad x, y \in M .$$

Upper estimate for the ungrounded kernel

We now estimate the remainder $G_{n/2,\alpha} - G^0_{n/2,\alpha}$. Choose $0 < \beta < \alpha$. For every $n \geqslant 2$ and suitable $C, C' > 0$,

$$0 \leqslant G_{n/2,\alpha}(x,y) - G^0_{n/2,\alpha}(x,y) := \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-\alpha t}\left(p_t(x,y) - p_t^0(x,y)\right) t^{n/2-1} dt$$
$$\leqslant C \int_0^\infty e^{-\beta t}\left(p_t(x,y) - p_t^0(x,y)\right) dt = C\left(G_{1,\beta} - G^0_{1,\beta}\right)(x,y)$$
$$\leqslant C \sup_{z \in \partial B} G_{1,\beta}(x,z) \leqslant C' \ .$$

Above, the second to last inequality follows from the maximum principle for local solutions to $(-\Delta_g + \beta)u = 0$, and the last inequality from the elliptic Harnack inequality for positive local solutions to $(-\Delta_g + \beta)u = 0$.

Bounds for the grounded kernel

The lower and upper bounds for the *grounded* resolvent kernel $\mathring{G}_\alpha$ for $\alpha > \lambda_*$ and then follow from the previous bounds and the fact that $\mathring{p}_t(x,y) = p_t(x,y) - \frac{1}{\mathrm{vol}(M)}$ and

$$\frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty e^{-\alpha t} t^{n/2-1} dt = \alpha^{-n/2} \ .$$

Bounds for all $\alpha > -\lambda_1$

In order to show the desired estimates for G_α in the whole range of $\alpha > -\lambda_1$, we use a perturbation argument based on the *resolvent identity*

$$\mathring{\mathsf{G}}_\alpha = \mathring{\mathsf{G}}_\beta + (\beta - \alpha)\,\mathring{\mathsf{G}}_\beta \mathring{\mathsf{G}}_\alpha, \tag{33}$$

valid for all $\beta > -\lambda_1$ and employed below for $\beta > \lambda_*$. By iteration, it follows that

$$\mathring{\mathsf{G}}_\alpha = \mathring{\mathsf{G}}_\beta \left(\sum_{\ell=0}^\infty \left((\beta - \alpha)\mathring{\mathsf{G}}_\beta\right)^\ell \right) .$$

The series is absolutely converging in $\mathrm{Lin}(\mathring{L}^2, \mathring{L}^2)$, since

$$\|(\beta - \alpha)\mathring{\mathsf{G}}_\beta\|_{\mathring{L}^2,\mathring{L}^2} \leqslant (\beta - \alpha)/(\beta + \lambda_1) < 1.$$

Let $\mathsf{T} = (\beta - \alpha) \sum_{\ell=0}^\infty \left((\beta - \alpha)\mathring{\mathsf{G}}_\beta\right)^\ell$. Then,

$$\mathring{\mathsf{G}}_\alpha^{n/2} = \mathring{\mathsf{G}}_\beta^{n/2}(\mathrm{Id} + \mathring{\mathsf{G}}_\beta \mathsf{T})^{n/2} = \mathring{\mathsf{G}}_{\beta/2}^{n/2}\left(1 + \sum_{k=1}^{n/2} \binom{n/2}{k} \mathring{\mathsf{G}}_\beta^k \mathsf{T}^k\right) = \mathring{\mathsf{G}}_\beta^{n/2} + \mathring{\mathsf{G}}_\beta^{n/2+1}\tilde{\mathsf{T}},$$

where $\tilde{\mathsf{T}} = \sum_{k=0}^{n/2-1} \binom{n/2}{k+1} \mathring{\mathsf{G}}_\beta^k \mathsf{T}^{k+1}$. Consequently, since all operators involved commute with each other,

$$\mathring{\mathsf{G}}_\alpha^{n/2} - \mathring{\mathsf{G}}_\beta^{n/2} = \underbrace{\mathring{\mathsf{G}}_\beta^{n/4+1/2}}_{\mathring{L}^2 \to \mathring{L}^\infty} \underbrace{\tilde{\mathsf{T}}}_{\mathring{L}^2 \to \mathring{L}^2} \underbrace{\mathring{\mathsf{G}}_\beta^{n/4+1/2}}_{\mathring{L}^1 \to \mathring{L}^2} .$$

Moreover, $\mathring{\mathsf{G}}_\beta^{n/4+1/2}$ is a bounded linear operator both from $\mathring{L}^1$ to $\mathring{L}^2$ and from $\mathring{L}^2$ to $\mathring{L}^\infty$. Indeed, for $u \in \mathring{L}^1$,

$$\begin{aligned}
\left\| \mathring{\mathsf{G}}_\beta^{n/4+1/2} u \right\|_{L^2}^2 &= \\
&= \int \left[\int \mathring{G}_{(n+2)/4,\beta}(x,y)\, u(y)\, \mathrm{dvol}(y) \int \mathring{G}_{(n+2)/4,\beta}(x,z)\, u(z)\, \mathrm{dvol}(z) \right] \mathrm{dvol}(x) \\
&= \iint \mathring{G}_{(n+2)/2,\beta}(y,z)\, u(y)\, u(z)\, \mathrm{dvol}(y) \mathrm{dvol}(z) \\
&\leqslant \sup_{y,z} \mathring{G}_{(n+2)/2,\beta}(y,z) \cdot \|u\|_{L^1}^2 \,,
\end{aligned}$$

and for $u \in \mathring{L}^2$,

$$\begin{aligned}
\|\mathring{\mathsf{G}}_\beta^{n/4+1/2} u\|_{L^\infty}^2 &= \sup_x \left(\int \mathring{G}_{(n+2)/4,\beta}(x,y) u(y) \mathrm{dvol}(y) \right)^2 \\
&\leqslant \sup_x \int \mathring{G}^2_{(n+2)/4,\beta}(x,y) \mathrm{dvol}(y) \cdot \int u^2(y)\, \mathrm{dvol}(y) \\
&= \sup_x \mathring{G}_{(n+2)/2,\beta}(x,x) \cdot \|u\|_{L^2}^2 \,.
\end{aligned}$$

Finiteness of both expressions is granted for $\beta > 0$ by Lemma 2.10. Thus, summarizing we obtain

$$\left\| \mathring{\mathsf{G}}_\alpha^{n/2} - \mathring{\mathsf{G}}_\beta^{n/2} \right\|_{\mathring{L}^1, \mathring{L}^\infty} < \infty \,.$$

Furthermore, consider $(\mathring{\mathsf{G}}_\alpha^{n/2} - \mathring{\mathsf{G}}_\beta^{n/2}) \circ \pi_g : L^1 \to L^\infty$, where as usual $\pi_g : u \mapsto u - \langle u \rangle_g$. Since π_g is the identity on $\mathring{L}^1$, by virtue of [32, Thm. 2.2.5], the operator $(\mathring{\mathsf{G}}_\alpha^{n/2} - \mathring{\mathsf{G}}_\beta^{n/2}) : \mathring{L}^1 \to L^\infty$ admits a bounded integral kernel. Therefore, $\mathring{\mathsf{G}}_\alpha^{n/2}$ admits an integral kernel with the same logarithmic divergence as $\mathring{\mathsf{G}}_\beta^{n/2}$. □

For completeness, we provide an estimate for the co-polyharmonic heat kernel that, in the case $n = 2$, reduces to the standard Gaussian estimate.

Remark 2.14 [85, Theorem 1.1]. Assume that the compact manifold (M, g) of even dimension n is a Lie group. Then, the co-polyharmonic heat semigroup $e^{-t\,\mathsf{P}_g}$ has an integral kernel (co-polyharmonic heat kernel), the modulus of which can be estimated by

$$|p_t(x,y)| \leqslant \frac{C_1}{t\wedge 1} \exp\left[-\left(\frac{d(x,y)^n}{C_2\, t}\right)^{\frac{1}{n-1}}\right] .$$

2.3 | Sobolev spaces and pairings

For $s \in \mathbb{R}_+$, we define the usual *Sobolev spaces* $\mathcal{H}^s_g := (1-\Delta_g)^{-\frac{s}{2}} L^2_g$ with norm $\|u\|_{\mathcal{H}^s_g} := \|(1-\Delta_g)^{\frac{s}{2}} u\|_{L^2_g}$, and $\mathcal{H}^{-s}_g$ as the completion of L^2_g w.r.t. the norm $\|u\|_{\mathcal{H}^s_g} := \|(1-\Delta_g)^{-\frac{s}{2}} u\|_{L^2_g}$ such that formally $\mathcal{H}^{-s}_g := (1-\Delta_g)^{\frac{s}{2}} L^2_g$.

For our purpose, however, it is more convenient to use slightly different Hilbert spaces defined in terms of the *normalized co-polyharmonic operator*

$$\mathsf{p}_g := a_n\, \mathsf{P}_g \tag{34}$$

with a_n as defined in (32), that is, $a_n := \frac{2}{\Gamma(n/2)\,(4\pi)^{n/2}}$.

We put

$$H^s_g := (1+\mathsf{p}_g)^{-\frac{s}{n}} L^2_g, \qquad \|u\|_{H^s_g} := \|(1+\mathsf{p}_g)^{\frac{s}{n}} u\|_{L^2_g}$$

and define H^{-s}_g as the completion of L^2_g w.r.t. the norm $\|u\|_{H^{-s}_g} := \|(1+\mathsf{p}_g)^{-\frac{s}{n}} u\|_{L^2_g}$. Moreover, we define the *grounded Sobolev spaces*

$$\mathring{H}^s_g := \mathsf{p}_g^{-\frac{s}{n}} \mathring{L}^2_g, \qquad \|u\|_{\mathring{H}^s_g} := \|\mathsf{p}_g^{\frac{s}{n}} u\|_{\mathring{L}^2_g}$$

and define $\mathring{H}^{-s}_g$ as the completion of $\mathring{L}^2_g$ w.r.t. the norm $\|u\|_{\mathring{H}^{-s}_g} := \|\mathsf{p}_g^{-\frac{s}{n}} u\|_{\mathring{L}^2_g}$.

The scalar product $\langle u|v\rangle_g := \int_M uv\, d\mathrm{vol}_g$ on L^2_g satisfies $|\langle u|v\rangle_g| \leqslant \|u\|_{H^{-s}_g} \cdot \|v\|_{H^s_g}$ for $u \in L^2_g$ and $v \in H^s_g$, and thus continuously extends to a bilinear form

$$\langle\,\cdot\,|\,\cdot\,\rangle_g \,:\, H^{-s}_g \times H^s_g \to \mathbb{R}$$

for every $s \geqslant 0$. For $u \in H^{-s}_g$, consistently with our previous definition for $u \in L^2_g$, we put

$$\langle u\rangle_g := \frac{1}{\mathrm{vol}_g(M)}\langle u|1\rangle_g\,, \qquad \pi_g(u) := u - \langle u\rangle_g\,.$$

Lemma 2.15. *For every admissible manifold* (M,g)*:*

(i) *The co-polyharmonic operator* P_g *is a compact perturbation of the poly-Laplacian: for every* $\alpha > -\lambda_1$*, there exists* $C_\alpha = C(\alpha,g) > 0$ *such that the operator* $\mathsf{S}_\alpha := \mathsf{P}_g - (\alpha-\Delta_g)^{n/2}$

satisfies

$$\left\langle \mathsf{S}_\alpha u \,\middle|\, u \right\rangle_{L^2_g} \leqslant C_\alpha \cdot \left\|(\alpha-\Delta)^{\frac{n-1}{4}} u\right\|^2_{L^2_g}, \qquad \forall u \in \mathcal{H}^{n/2}. \tag{35}$$

In particular, $\langle \mathsf{S}_1 u \,|\, u\rangle_{L^2_g} \leqslant C_1 \, \|u\|^2_{\mathcal{H}^{(n-1)/2}}$.

(ii) *For every* $s \in \mathbb{R}$*, the spaces* $\mathcal{H}^s_g$ *and* H^s_g *coincide as sets and the respective norms are bi-Lipschitz equivalent to each other.*

(iii) *The operator* p_g *with domain* H^n_g *has discrete spectrum* $\operatorname{spec}(\mathsf{p}_g) = \left\{\nu_j\right\}_{j\in\mathbb{N}_0}$*, indexed with multiplicities, satisfying* $\nu_j \geqslant 0$ *for all* j*, and* $\nu_0 = 0$ *with multiplicity 1. The corresponding family of eigenfunctions* $\left(\psi_j\right)_{j\in\mathbb{N}_0}$ *forms an orthonormal basis of* L^2_g*.*

(iv) *For every* $s \in \mathbb{R}$*,*

$$u \in \mathring{H}^s_g \quad \Longleftrightarrow \quad u = \sum_{j\geqslant 1} \alpha_j \psi_j \text{ with } \sum_{j\geqslant 1} \nu_j^{\frac{2s}{n}} \alpha_j^2 < \infty$$

and

$$u \in H^s_g \quad \Longleftrightarrow \quad u = \sum_{j\geqslant 0} \alpha_j \psi_j \text{ with } \sum_{j\geqslant 0} (1+\nu_j)^{\frac{2s}{n}} \alpha_j^2 < \infty.$$

Hence, in particular, $\mathring{H}^s_g = \{u \in H^s_g \,:\, \langle u\rangle_g = 0\}$*.*

(v) *For every* $s > 0$*, there exists* $C = C(s)$ *such that for all* $r \in \mathbb{R}$*,*

$$\mathring{H}^{r+s}_g \subset \mathring{H}^r_g, \qquad \|\cdot\|_{\mathring{H}^r_g} \leqslant C \, \|\cdot\|_{\mathring{H}^{r+s}_g}.$$

(vi) *For every* $r \in \mathbb{R}$*, the bounded operator* $\mathsf{p}_g : \mathring{H}^{n+r}_g \to \mathring{H}^r_g$ *has bounded inverse* $\mathsf{k}_g : \mathring{H}^r_g \to \mathring{H}^{n+r}_g$*. For* $r = 0$*, the operator* $\mathsf{k}_g : \mathring{L}^2_g \to \mathring{L}^2_g$ *admits a unique nonrelabeled extension* $\mathsf{k}_g : L^2_g \to \mathring{L}^2_g$*, vanishing on constants and satisfying* $\mathsf{k}_g \mathsf{p}_g = \pi_g$ *on* L^2_g*. This extension is an integral operator on* L^2_g *with symmetric kernel*

$$k_g(x,y) := \sum_{j=1}^{\infty} \frac{\psi_j(x)\,\psi_j(y)}{\nu_j}\,, \qquad x, y \in M\,, \tag{36}$$

where the convergence of the series is understood in $L^2_g \otimes L^2_g$*.*

(vii) *For* $\ell \in \mathbb{N}$*, define the operators* $\mathsf{k}_{g,\ell} : L^2_g \to \mathring{L}^2_g$ *by*

$$\mathsf{k}_{g,\ell}\, u := \sum_{j=1}^{\ell} \frac{\left\langle \psi_j \,\middle|\, u\right\rangle_{L^2}}{\nu_j} \, \psi_j\,.$$

Then, for every $u \in L^2_g$*, as* $\ell \to \infty$*,*

$$\mathsf{k}_{g,\ell}\, u \longrightarrow \mathsf{k}_g u \quad \text{in } L^\infty\,. \tag{37}$$

(viii) *The spectrum of* p_g *satisfies the Weyl asymptotic. With* $N(\nu)$ *the number of eigenvalues lower than* ν*, we get* $N(\nu) = c\,\nu + O(\nu^{1-1/n})$ *as* $\nu \to \infty$*. In particular, we find*

$$\nu_j = cj + O(j^{1-1/n}) , \qquad j \to \infty . \tag{38}$$

(ix) *Given any* $r \in \mathbb{R}$*, the embedding* $\mathrm{Id} : \mathring{H}^{r+s}_g \hookrightarrow \mathring{H}^r_g$ *is trace class if and only if* $s > n$*, and it is Hilbert–Schmidt if and only if* $s > \frac{n}{2}$*.*

Proof.

(i) For every $\alpha > -\lambda_1$, the operator S_α is a linear differential operator of order $\leqslant n-1$ with smooth (hence bounded) coefficients on M, and (35) readily follows.

(ii) It suffices to show the statement for $s = 1/2$. The claim for any other $s > 0$ then follows by spectral calculus, and for $s < 0$ by duality.

As a consequence of Theorem 1.3 (iv) and admissibility, the (strictly) positive operator $(\mathsf{p}_g, \mathring{\mathcal{H}}^n_g)$ has positive self-adjoint square root $(\sqrt{\mathsf{p}_g}, \mathring{\mathcal{H}}^{n/2}_g)$, and the latter defines a Hilbert norm on $\mathring{\mathcal{H}}^{n/2}_g$. Thus, the linear operator $\iota := (-\Delta_g)^{-n/4}\sqrt{\mathsf{p}_g} : \mathring{\mathcal{H}}^{n/2}_g \to \mathring{\mathcal{H}}^{n/2}_g$ is well defined, positive, and injective. Moreover, ι is an isometry

$$\iota \colon \left(\mathring{\mathcal{H}}^{n/2}_g, \|\sqrt{\mathsf{p}_g}\cdot\|_{\mathring{\mathcal{H}}_g}\right) \longrightarrow \left(\mathring{\mathcal{H}}^{n/2}_g, \|\cdot\|_{\mathring{\mathcal{H}}^{n/2}_g}\right) ,$$

and, in fact, unitary, since $\ker\iota = \{0\}$ by strict positivity of both $\sqrt{\mathsf{p}_g}$ and $(-\Delta_g)^{-n/4}$ on the appropriate spaces of grounded functions. As a consequence, $\iota \colon \mathring{\mathcal{H}}^{n/2}_g \to \mathring{\mathcal{H}}^{n/2}_g$ is surjective, and thus bijective. It suffices to show that it is also $\mathring{\mathcal{H}}^{n/2}_g$-bounded, in which case it has an $\mathring{\mathcal{H}}^{n/2}_g$-bounded inverse ι^{-1} by the bounded inverse theorem. The former fact follows if we show that $\iota\iota^*$ is $\mathring{\mathcal{H}}^{n/2}_g$-bounded. We have

$$\iota\iota^* = (-\Delta_g)^{-n/4}\mathsf{P}_g(-\Delta_g)^{-n/4} = \mathrm{Id}_{\mathring{\mathcal{H}}^{n/2}_g} + (-\Delta_g)^{-n/4}\mathsf{S}_0(-\Delta_g)^{-n/4} .$$

By squaring the operators in (35) with $\alpha = 0$, the latter is a $\mathring{\mathcal{H}}^{n/2}_g$-bounded perturbation of the identity on $\mathring{\mathcal{H}}^{n/2}_g$, and the assertion follows.

(iii) Since $\mathring{H}^n_g$ embeds compactly into $\mathring{H}^0_g$ by the Rellich–Kondrashov Theorem, the operator $\mathsf{k}_g : \mathring{H}^0_g \to \mathring{H}^0_g$ is compact, being the composition of the bounded operator $\mathsf{k}_g : \mathring{H}^0_g \to \mathring{H}^n_g$ with the compact Sobolev embedding. The spectral properties follow from the (strict) positivity of $(p_g, \mathring{H}^n)$ on $\mathring{H}^0_g$ and the $\mathring{H}^0_g$-compactness of k_g. The assertion on eigenfunctions holds by the spectral theorem for unbounded self-adjoint operators.

(iv) Direct calculation and the fact that $\inf_{j\geqslant 1} \frac{\nu_j}{1+\nu_j} > 0$.

(v) Let us first observe that $\|(1-\Delta_g)^{1/2}u\|_{L^2_g} \geqslant \frac{1}{C}\,\|u\|_{L^2_g}$ for all $u \in L^2_g$, and thus,

$$\|(1-\Delta_g)^{s/2}u\|_{L^2_g} \geqslant \frac{1}{C^s}\,\|u\|_{L^2_g}$$

for all $s > 0$. By positivity of p_g on $\mathring{L}^2_g$ and norm equivalence of $\mathcal{H}^n_g$ and H^n_g, it follows

$$\|\mathsf{p}_g u\|_{\mathring{L}^2_g} \geqslant \frac{1}{C'} \|(1+\mathsf{p}_g)u\|_{\mathring{L}^2_g} \geqslant \frac{1}{C''} \|(1-\Delta_g)^{n/2} u\|_{\mathring{L}^2_g} \geqslant \frac{1}{C'''} \|u\|_{\mathring{L}^2_g}$$

for $u \in \mathring{L}^2_g$. This lower estimate for the self-adjoint operator p_g implies an analogous estimate for any of its positive powers. Thus,

$$\|u\|_{\mathring{H}^s_g} = \|\mathsf{p}_g^{s/n} u\|_{\mathring{L}^2_g} \geqslant \frac{1}{C_s} \|u\|_{\mathring{L}^2_g}$$

for any $s > 0$. This proves the claim for $r = 0$. The claim for general r follows readily.

(vi) It suffices to show the statement for $r = 0$. We show that $\sqrt{\mathsf{p}_g} : \mathring{H}^{n/2}_g \to \mathring{H}^0_g$ is invertible with bounded inverse $\sqrt{\mathsf{k}_g}$. As a consequence of the bijectivity of ι in (ii), and since $(-\Delta_g)^{n/2} : \mathring{H}^{n/2}_g \to \mathring{H}^0_g$ is surjective, the operator $\sqrt{\mathsf{p}_g} = (-\Delta_g)^{n/2}\iota : \mathring{H}^{n/2}_g \to \mathring{H}^0_g$ is as well surjective, and thus bijective. Its inverse $\sqrt{\mathsf{k}_g} := \iota^{-1}(-\Delta_g)^{-n/2} : \mathring{H}^0_g \to \mathring{H}^{n/2}_g$ is a bounded operator, since so are $\iota^{-1} : \mathring{H}^{n/2}_g \to \mathring{H}^{n/2}_g$, by (ii), and $(-\Delta_g)^{-n/2} : \mathring{H}^0_g \to \mathring{H}^{n/2}_g$.

(vii) By the norm equivalence stated in (ii),

$$\left\|\mathsf{k}_g u - \mathsf{k}_{g,\ell}\, u\right\|^2_{\mathcal{H}^n_g} \simeq \left\|\mathsf{p}_g(\mathsf{k}_g u - \mathsf{k}_{g,\ell}\, u)\right\|^2_{L^2_g} = \sum_{j=\ell+1}^{\infty} \langle \psi_j \,|\, u\rangle^2_{L^2_g} \longrightarrow 0$$

as $\ell \to \infty$ for every $u \in L^2_g$. Hence, by Sobolev embedding, $\mathsf{k}_{g,\ell}\, u \to \mathsf{k}_g u$ in L^∞.

(viii) is Hörmander's Weyl law for positive pseudodifferential operators. Indeed, choosing $dx = d\mathsf{vol}_g$ and integrating [51, Eqn. (1.1)], the assertion follows from [51, Thm. 1.1].

(ix) For any $s > 0$, the embedding $\mathrm{Id} : \mathring{H}^{r+s}_g \hookrightarrow \mathring{H}^r_g$ is trace class (or Hilbert–Schmidt, resp.) if and only if the operator $\mathsf{p}_g^{-s/n} : \mathring{L}^2_g \to \mathring{L}^2_g$ is so. By definition, the latter is trace class (or Hilbert–Schmidt, resp.) if and only if

$$\sum_j \nu_j^{-s/n} < \infty \qquad \left(\text{or } \sum_j \nu_j^{-2s/n} < \infty, \text{resp.}\right)$$

which, in turn — according to (viii) — holds true if and only if $s > n$ (or $s > n/2$, resp.). □

Remark 2.16.

(a) Elliptic regularity theory implies that off the diagonal of $M \times M$, the function $(x, y) \mapsto k_g(x, y)$ is C^∞.

(b) The symmetry of the integral kernel k_g implies that

$$\int_M k_g(x, y)\, d\mathsf{vol}_g(y) = 0 \,, \qquad x \in M \,. \tag{39}$$

Indeed, $\mathsf{k}_g f \in \mathring{L}^2_g$ implies $\int \left[\int k_g(x, y) d\mathsf{vol}_g(x)\right] f(y)\, d\mathsf{vol}_g(y) = 0$ for all $f \in \mathring{L}^2_g$ that, in turn, implies that $\int k_g(x, y) d\mathsf{vol}_g(x)$ is constant in y. By symmetry, this constant must vanish.

Of particular importance in the sequel will be the spaces H^s_g for $s = \frac{n}{2}$ and $s = -\frac{n}{2}$.

Definition 2.17. Given any admissible manifold (M, g), we denote the scalar products for the Hilbert spaces $H_g^{n/2}$ and $H_g^{-n/2}$ by

$$\mathfrak{p}_g(u, v) := \int \sqrt{\mathsf{p}_g} u \sqrt{\mathsf{p}_g} v \,\mathrm{d}\,\mathrm{vol}_g , \qquad u,\, v \in H_g^{n/2} , \tag{40}$$

$$\mathcal{K}_g(u, v) := \int \sqrt{\mathsf{k}_g} u \sqrt{\mathsf{k}_g} v \,\mathrm{d}\,\mathrm{vol}_g , \qquad u,\, v \in \mathring{H}_g^{-n/2} . \tag{41}$$

Restricted to the space $\mathring{L}_g^2$, the bilinear form $\mathcal{K}_g$ is given by

$$\mathcal{K}_g(u, v) = \langle \mathsf{k}_g u \,|\, v \rangle_{L_g^2} = \iint u(x)\, k_g(x, y)\, v(y)\, \mathrm{dvol}_g(x)\, \mathrm{dvol}_g(y) . \tag{42}$$

Observe that the right-hand side here is also well defined for ungrounded $u, v \in L_g^2$ that allows us to consider $\mathcal{K}_g$ also as a bilinear form on L_g^2 with $\mathcal{K}_g(u + C, v + C') = \mathcal{K}_g(u, v)$ for $u, v \in \mathring{L}^2(M, \mathrm{vol}_g)$ and $C, C' \in \mathbb{R}$. Moreover, we always implicitly extend the operator k_g to L_g^2 by setting $\mathsf{k}_g c = 0$ for all constant c. It is the *pseudoinverse* of p_g on L_g^2 in the sense that:

$$\mathsf{k}_g \mathsf{p}_g = \mathsf{p}_g \mathsf{k}_g = \pi_g .$$

We call k_g the *co-polyharmonic Green operator*. It has the integral kernel k_g given in (36), and we call k_g the *co-polyharmonic Green kernel*.

In the following lemma, we make use of some arguments in complex-interpolation theory. We refer the reader to [86, §1.2.1] for the necessary standard definitions of interpolation couple, and to [86, §1.9.2-3] for results on complex interpolation. We denote by $[A_0, A_1]_\theta$, $\theta \in (0, 1)$, the standard complex interpolation of Banach spaces A_0, A_1.

For Banach spaces A_0, A_1, A of functions on M, we write

$$\cdot : A_0 \times A_1 \longrightarrow A$$

to indicate that the pointwise product on $A_0 \times A_1$ is a continuous bilinear map with range contained in A, that is, for every $f \in A_0$ and $g \in A_1$, we have $f g \in A$ and $\|f g\|_A \lesssim \|f\|_{A_0} \|g\|_{A_1}$.

Lemma 2.18.

(i) *For every $s \in [0, \frac{n}{2}]$, for every $\varepsilon > 0$,*

$$\cdot : \mathcal{H}_g^s \times \mathcal{H}_g^{n/2+\varepsilon} \longrightarrow \mathcal{H}_g^s .$$

(ii) *For every $r, s, t \in \mathbb{R}$ with $s, t \geqslant r \geqslant 0$ and $s + t \geqslant r + \frac{n}{2}$,*

$$\cdot : \mathcal{H}_g^s \times \mathcal{H}_g^t \longrightarrow \mathcal{H}_g^r . \tag{43}$$

(iii) *For every $t \geqslant \frac{n}{2}$, for every $s \in \mathbb{R}$,*

$$\cdot : \mathcal{H}_g^s \times \mathcal{H}_g^t \longrightarrow \mathcal{H}_g^s . \tag{44}$$

(iv) *For every $s \in \mathbb{R}$, for every $\varphi \in C^\infty(M)$,*

$$\langle e^{n\varphi} u | v \rangle_g = \langle u | e^{n\varphi} v \rangle_g = \langle u | v \rangle_{e^{2\varphi} g} , \qquad u \in \mathcal{H}^{-s}_g, v \in \mathcal{H}^s_g . \tag{45}$$

(v) *All the above assertions hold with H^s_g in place of $\mathcal{H}^s_g$.*

Proof. For (i) and (ii), we adapt to our setting the proof for Euclidean spaces in [5, Lem. 5.2, Thm. 5.1].

(i) Since $\frac{n}{2} + \varepsilon > \frac{n}{2}$, by [20, Thm. 24], the space $\mathcal{H}^{n/2+\varepsilon}_g$ is an algebra and

$$\cdot : \mathcal{H}^{n/2+\varepsilon}_g \times \mathcal{H}^{n/2+\varepsilon}_g \longrightarrow \mathcal{H}^{n/2+\varepsilon}_g . \tag{46}$$

Combing [87, Thm. 5(iii), Thm. 2(iii), Thm. 4(i)], we have $\mathcal{H}^{n/2+\varepsilon}_g \hookrightarrow L^\infty_g$. Thus, since $\cdot : L^\infty_g \times L^2_g \to L^2_g$, we further have

$$\cdot : \mathcal{H}^{n/2+\varepsilon}_g \times \mathcal{H}^0_g \longrightarrow \mathcal{H}^0_g . \tag{47}$$

By complex interpolation of bilinear forms (see [86, §1.19.5]), the pointwise product in (46) and (47) interpolates to

$$\cdot : \mathcal{H}^{n/2+\varepsilon}_g \times [\mathcal{H}^0_g, \mathcal{H}^{n/2+\varepsilon}_g]_\theta \longrightarrow [\mathcal{H}^0_g, \mathcal{H}^{n/2+\varepsilon}_g]_\theta , \qquad \theta \in (0,1). \tag{48}$$

Choosing θ so that $s = \theta(\frac{n}{2} + \varepsilon)$, we have $[\mathcal{H}^0_g, \mathcal{H}^{n/2+\varepsilon}_g]_\theta = \mathcal{H}^s_g$ by [82, Cor. 4.6], and the conclusion follows.

(ii) If $r > \frac{n}{2}$, the space $\mathcal{H}^r_g$ is an algebra, and therefore, since $s, t \geqslant r$,

$$\cdot : \mathcal{H}^s_g \times \mathcal{H}^t_g \hookrightarrow \mathcal{H}^r_g \times \mathcal{H}^r_g \longrightarrow \mathcal{H}^r_g ,$$

which is the assertion. If otherwise $r \in [0, \frac{n}{2}]$, let $\varepsilon := s + t - r - \frac{n}{2} > 0$. By (i),

$$\mathcal{H}^r_g \times \mathcal{H}^{n/2+\varepsilon}_g \longrightarrow \mathcal{H}^r_g , \tag{49a}$$

$$\mathcal{H}^{n/2+\varepsilon}_g \times \mathcal{H}^r_g \longrightarrow \mathcal{H}^r_g . \tag{49b}$$

Now, since $r \leqslant s$, we have $s \leqslant s + t - r$ and so $s \leqslant \frac{n}{2} + \varepsilon$. Thus, there exists $\theta \in [0,1]$ with $(1-\theta)r + \theta(\frac{n}{2} + \varepsilon) = s$ and $(1-\theta)(\frac{n}{2} + \varepsilon) + \theta r = t$. Again, by [82, Cor. 4.6], we have $[\mathcal{H}^r_g, \mathcal{H}^{n/2+\varepsilon}_g]_\theta = \mathcal{H}^s_g$ and $[\mathcal{H}^{n/2+\varepsilon}_g, \mathcal{H}^r_g]_\theta = \mathcal{H}^t_g$, and (43) follows by complex interpolation of the pointwise product in (49a), (49b).

(iii) If $s \geqslant 0$, the assertion is (ii) with $r := s$. If $s < 0$, we argue as follows. For every $f \in \mathcal{H}^t_g$ and $v \in \mathcal{H}^{-s}_g$, we have $fv \in \mathcal{H}^{-s}_g$ by (ii). Thus, for every $u \in L^2_g$, for some constant $C = C_{s,t} > 0$,

$$\left\langle fu \,\middle|\, v \right\rangle_{L^2_g} = \left\langle u \,\middle|\, fv \right\rangle_{L^2_g} \leqslant \|u\|_{\mathcal{H}^s_g} \|fv\|_{\mathcal{H}^{-s}_g} \leqslant C_{s,t} \|u\|_{\mathcal{H}^s_g} \|v\|_{\mathcal{H}^{-s}_g} .$$

As a consequence, fu defines a continuous linear functional on $\mathcal{H}^{-s}_g$ and we have

$$\|fu\|_{\mathcal{H}^s_g} \leqslant C_{s,t} \|f\|_{\mathcal{H}^t_g} \|u\|_{\mathcal{H}^s_g} , \qquad u \in L^2_g .$$

By density of L^2_g in $\mathcal{H}^s_g$, the above inequality extends to $u \in \mathcal{H}^s_g$, which proves the assertion.

(iv) follows easily from (iii) by approximation of $u \in \mathcal{H}^{-s}_g$ by $u_j \in L^2_g$.

(v) is an immediate consequence of Lemma 2.15 (ii). □

2.4 | Estimates for co-polyharmonic Green kernels

Theorem 2.19. *For every admissible manifold (M, g), the co-polyharmonic Green kernel k_g satisfies*

$$\left| k_g(x,y) - \log \frac{1}{d_g(x,y)} \right| \leqslant C_0 \tag{50}$$

for some $C_0 = C_0(g)$.

Proof. By the second resolvent identity for the operators k_g and $\frac{1}{a_n}\mathring{\mathsf{G}}_{n/2} : \mathring{H}^0_g \to \mathring{H}^0_g$,

$$\mathsf{k}_g - \frac{1}{a_n}\mathring{\mathsf{G}}_{n/2} = \mathsf{k}_g\, \mathsf{S}_0\, \mathring{\mathsf{G}}_{n/2} = \mathsf{k}_g\, \mathsf{S}_0\, \mathring{\mathsf{G}}_{\frac{n-1}{4}}\, \mathring{\mathsf{G}}_{\frac{n+1}{4}} \tag{51}$$

with $\mathsf{S}_0 = \mathsf{P}_g - (-\Delta_g)^{n/2}$ as in Lemma 2.15(i). Similarly to the proof of Proposition 2.13, the operators $\mathring{\mathsf{G}}_{\frac{n+1}{4}} : \mathring{L}^1 \to \mathring{H}^0$ and $\mathring{\mathsf{G}}_{\frac{n-1}{4}} : \mathring{H}^0 \to \mathring{H}^{\frac{n-1}{2}}$ are bounded. By Theorem 1.3(iii), S_0 is a differential operator of order at most $n-1$ with smooth (hence bounded) coefficients. As a consequence, $\mathsf{S}_0 : \mathring{H}^{\frac{n-1}{2}} \to \mathring{H}^{-\frac{n-1}{2}}$ is a bounded operator. Furthermore, choosing $r = -\frac{n-1}{2}$ in Lemma 2.15(ii), the operator $\mathsf{k}_g : \mathring{H}^{-\frac{n-1}{2}} \to \mathring{H}^{\frac{n+1}{2}}$ is bounded.

Combining the previous assertions with (51) shows that $\mathsf{k}_g - \frac{1}{a_n}\mathring{\mathsf{G}}_{n/2} : \mathring{L}^1 \to \mathring{H}^{\frac{n+1}{2}}$ is bounded; thus, $\mathsf{k}_g - \frac{1}{a_n}\mathring{\mathsf{G}}_{n/2} : \mathring{L}^1 \to L^\infty$ is bounded as well, by continuity of the Sobolev–Morrey embedding. Finally, by [32, Thm. 2.2.5], the latter operator admits a bounded integral kernel, and the conclusion follows from Proposition 2.13. □

The previous theorem has also been derived with different (and partly rather sketchy) arguments in [66, Lemma 2.1].

Proposition 2.20. *Assume that (M, g) is admissible and that $g' := e^{2\varphi} g$ for some $\varphi \in C^\infty(M)$. Then the co-polyharmonic Green operator $\mathsf{k}_{g'}$ is given by*

$$\mathsf{k}_{g'} u = \left(\pi_{g'} \circ \mathsf{k}_g\right)(e^{n\varphi} u)\,, \qquad u \in L^2\,, \tag{52}$$

and the co-polyharmonic Green kernel $k_{g'}$ by

$$k_{g'}(x,y) = k_g(x,y) - \frac{1}{2}\bar\phi(x) - \frac{1}{2}\bar\phi(y) \tag{53}$$

with $\bar\phi \in C^\infty(M)$ defined by

$$\bar\phi := \frac{2}{\mathrm{vol}_{g'}(M)} \int k_g(\,\cdot\,, z)\, d\mathrm{vol}_{g'}(z) - \frac{1}{\mathrm{vol}_{g'}(M)^2} \iint k_g(z,w)\, d\mathrm{vol}_{g'}(z)\, d\mathrm{vol}_{g'}(w)\,.$$

Proof. Let $k_{g'}$ be the integral kernel defined by the right-hand side of (53). Obviously, $k_{g'}$ is symmetric. Furthermore, by (50),

$$\left|k_{g'}(x,y) - k_g(x,y)\right| \leqslant \frac{3}{\mathrm{vol}_{g'}(M)} \sup_w \int |k_g(z,w)| d\mathrm{vol}_{g'}(z)$$

$$\leqslant 3\,C\,\mathrm{vol}_{g'}(M) + \frac{3}{\mathrm{vol}_{g'}(M)} \sup_w \int \left|\log \frac{1}{d_g(z,w)}\right| d\mathrm{vol}_{g'}(z) < \infty\ , \tag{54}$$

and, since $e^{\inf\varphi}\, d_g \leqslant d_{g'} \leqslant e^{\sup\varphi}\, d_g$,

$$\left|\log\frac{1}{d_g(x,y)} - \log\frac{1}{d_{g'}(x,y)}\right| \leqslant C_{\varphi,g}\ . \tag{55}$$

Thus, the kernel $k_{g'}$ satisfies (50) with $k_{g'}$ in place of k_g and g' in place of g for some constant $C_0(g')$.

Moreover, straightforward calculation yields the identity (52) for the integral operator $\mathsf{k}_{g'}$ associated with the kernel $k_{g'}$. It remains to prove that the operator $\mathsf{k}_{g'}$ is the inverse of $\mathsf{p}_{g'}$. To see this, recall that we have $\mathsf{p}_{g'} = e^{-n\varphi}\mathsf{p}_g$. Thus, for all $u \in \mathring{H}^{n/2}_{g'}$,

$$\mathsf{k}_{g'}\,\mathsf{p}_{g'}u = \mathsf{k}_g\mathsf{p}_g u - \left\langle \mathsf{k}_g\mathsf{p}_g u\right\rangle_{g'} = u - \langle u\rangle_g - \left\langle u - \langle u\rangle_g\right\rangle_{g'} = u.$$

Consequently, we have $\mathsf{k}_{g'}\,\mathsf{p}_{g'}u = u = \mathsf{p}_{g'}\,\mathsf{k}_{g'}u$ and the claim follows by uniqueness of the inverse. □

Remark 2.21. The transformation formula (53) for the co-polyharmonic Green kernels can be rephrased as follows. Given $\varphi \in \mathcal{C}^\infty(M)$, let $\varphi_0 := \varphi - c$ with c chosen such that $\int e^{n\varphi_0} d\mathrm{vol} = 1$. Then,

$$k_{e^{2\varphi}g}(x,y) = k_g(x,y) - \mathsf{k}_g(e^{n\varphi_0})(x) - \mathsf{k}_g(e^{n\varphi_0})(y) + \left\langle \mathsf{k}_g(e^{n\varphi_0}), e^{n\varphi_0}\right\rangle_{L^2_g}\ . \tag{56}$$

Proposition 2.22. *Given any admissible manifold* (M,g) *and* $g' = e^{2\varphi}g$ *with* $\varphi \in \mathcal{C}^\infty(M)$*, then*

(i) $u \in \mathring{H}^{n/2}_g$ *implies* $u \in \mathring{H}^{n/2}_{g'}$ *and* $\|u\|_{\mathring{H}^{n/2}_{g'}} = \|u\|_{\mathring{H}^{n/2}_g}$ *or, in other words,*

$$\mathfrak{p}_{g'}(u,u) = \mathfrak{p}_g(u,u). \tag{57}$$

(ii) $u \in \mathring{H}^{-n/2}_g$ *implies* $e^{-n\varphi}u \in \mathring{H}^{-n/2}_{g'}$ *and* $\|e^{-n\varphi}u\|_{\mathring{H}^{n/2}_{g'}} = \|u\|_{\mathring{H}^{n/2}_g}$ *or, in other words,*

$$\mathcal{K}_{g'}(e^{-n\varphi}u, e^{-n\varphi}u) = \mathcal{K}_g(u,u). \tag{58}$$

Proof.

(i) Immediate consequence of $\mathsf{P}_{g'}u = e^{-n\varphi}\mathsf{P}_g$ and $\mathrm{vol}_{g'} = e^{n\varphi}\mathrm{vol}_g$.
(ii) The norm identity follows from (52) and (2.18)(v):

$$\mathcal{K}_{g'}(e^{-n\varphi}u, e^{-n\varphi}u) = \langle e^{-n\varphi}u|\mathsf{k}_{g'}(^{-n\varphi}u)\rangle_{g'} = \langle e^{-n\varphi}u|\pi_{g'}(\mathsf{k}_g u)\rangle_{g'}$$

$$= \langle u|\pi_{g'}(\mathsf{k}_g u)\rangle_g = \langle u|\mathsf{k}_g u\rangle_g - \langle u|1\rangle_g \cdot \langle \mathsf{k}_g u\rangle_{g'} = \mathcal{K}_g(u,u).$$

Moreover, $\langle u\rangle_g = 0$ if and only if $\langle e^{-n\varphi}u\rangle_{g'} = 0$. □

3 | THE CO-POLYHARMONIC GAUSSIAN FIELD

In what follows, we consider an admissible manifold (M, g) of even dimension n. We make use of the normalized co-polyharmonic operator $\mathsf{p}_g := a_n \mathsf{P}_g$ and its inverse k_g with symmetric integral kernel k_g that has precise logarithmic divergence

$$\left| k_g(x, y) - \log \frac{1}{d_g(x, y)} \right| \leqslant C, \qquad \forall x, y \in M. \tag{59}$$

Our goal is to define and analyze a random field h on M with law formally characterized as

$$d\boldsymbol{\nu}_g(h) = \frac{1}{Z_g} \exp\left(-\frac{1}{2}\mathfrak{p}_g(h, h)\right) dh \tag{60}$$

where dh stands for the (nonexisting) uniform distribution on fields and Z_g denotes some normalization constant.

3.1 | Existence and uniqueness, equivalent characterizations

Definition 3.1. A co-polyharmonic Gaussian field on (M, g) is a centered Gaussian random variable h on $\mathring{H}_g^{-s}$ for some $s > 0$ with covariance

$$\mathbf{E}\left[\langle h|u\rangle_g \cdot \langle h|v\rangle_g\right] = \mathcal{K}_g(u, v) \qquad \forall u, v \in \mathring{H}_g^s . \tag{61}$$

Here, $\langle \cdot | \cdot \rangle_g$ on the left-hand side denotes the pairing $\langle \cdot | \cdot \rangle_{\mathring{H}_g^{-s}, \mathring{H}_g^s}$, and the right-hand side can be rewritten as

$$\mathcal{K}_g(u, v) = \iint k_g(x, y)\, u(x)\, v(y)\, \mathrm{dvol}_g(x)\, \mathrm{dvol}_g(y) = \left\langle u \,\middle|\, v \right\rangle_{\mathring{H}_g^{-n/2}}. \tag{62}$$

Theorem 3.2. *For every admissible manifold (M, g), there exists a co-polyharmonic Gaussian field, unique in distribution.*

More precisely, for any $s > 0$, there exists an $\mathring{H}_g^{-s}$-valued co-polyharmonic Gaussian field that is unique in distribution. It is supported on

$$\bigcap_{t>0} \mathring{H}_g^{-t}.$$

For all $s, t > 0$, an $\mathring{H}_g^{-s}$-valued co-polyharmonic Gaussian field and an $\mathring{H}_g^{-t}$-valued co-polyharmonic Gaussian field coincide in distribution.

We denote the law of *the* co-polyharmonic Gaussian field by $\mathsf{CGF}_{M,g}$ or simply — since throughout this paper, mostly M is fixed — by CGF_g. Equivalently, CGF_g can be characterized as the unique centered Gaussian probability measure $\boldsymbol{\nu}_g$ on $\mathring{H}_g^{-s}$ that satisfies

$$\int e^{i\left\langle h \,\middle|\, u \right\rangle_g} d\boldsymbol{\nu}_g(h) = \exp\left[-\tfrac{1}{2}\mathcal{K}_g(u, u)\right] \qquad \forall u \in \mathring{H}_g^s . \tag{63}$$

Proof. Let us consider the *abstract Wiener space* (ι, H, B) in the sense of Gross, following the notation and presentation in [8], with the Banach space $B := \mathring{H}_g^{-\varepsilon}$ ("Wiener space"), the Hilbert space $H := \mathring{H}_g^{n/2}$ ("Cameron-Martin space"), and the embedding $\iota : H \hookrightarrow B$ that is "measurable in the sense of Gross," cf. [8, Example 3.9.7], since

$$\|u\|_B = \|Tu\|_H$$

with the operator $T = \mathsf{p}_g^{-1-\varepsilon/n}$ being Hilbert–Schmidt according to Lemma 2.15 (ix). The existence of the requested Gaussian measure on B is then a key result of the theory of abstract Wiener spaces [8, Theorem 3.9.5]. □

Remark 3.3. Based on the Bochner–Minlos theorem, the co-polyharmonic Gaussian field can alternatively be defined as a random field on the space $\mathfrak{D}'$ of distributions on M. Here, $\mathfrak{D} := C^\infty(M)$ denotes the space of test functions, endowed with its usual Fréchet topology, which is a nuclear space, see, for instance, the comments preceding [43, Ch. II, Thm. 10, p. 55]. $\mathfrak{D}'$, the topological dual of $\mathfrak{D}$, is endowed with the Borel σ-algebra induced by the weak* topology. Set $\chi(u) := \exp[-\frac{1}{2}\mathcal{K}_g(u,u)]$. It satisfies $\chi(0) = 1$. Moreover, since M is admissible, $\mathcal{K}_g$ is a semidefinite inner product on $\mathfrak{D}$, thus, by, for example, [61, Prop. 2.4], χ is totally positive definite. By Lemma 2.15(ii), $u \mapsto \sqrt{\mathcal{K}_g(u,u)}$ is continuous with respect to the $\mathcal{H}_g^{-n/2}$-norm on $\mathfrak{D}$. Since $\mathfrak{D}$ embeds continuously into $\mathcal{H}_g^{-s}$ for every $s \in \mathbb{R}$, the functional χ is continuous on $\mathfrak{D}$. The claim follows by the Bochner–Minlos Theorem [89, §IV.4.3, Thm. 4.3, p. 410].

In the sequel, we will freely switch between the general representation of the co-polyharmonic Gaussian field as a measurable map $h : \Omega \to \mathring{H}_g^{-s}$ with law $h_*\mathbf{P} = \mathsf{CGF}_{g)}$, defined on some probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, and the standard representation of the co-polyharmonic Gaussian where $\Omega = \mathring{H}_g^{-s}$ and $h = \mathrm{Id}$.

Our definition implies that a co-polyharmonic Gaussian field h is *grounded*, in the sense that $\langle h|c\rangle_g = 0$ for all constant c.

Remark 3.4. An $\mathring{H}_g^{-\varepsilon}$-valued centered Gaussian random field h is a co-polyharmonic Gaussian field on (M, g) if and only if $\xi := \sqrt{\mathsf{p}_g}\, h$ is a *grounded white noise on* (M, g), that is, a $\mathring{H}_g^{-n/2-\varepsilon}$-valued centered Gaussian random field with covariance

$$\mathbf{E}\big[\langle \xi|u\rangle_g \langle \xi|v\rangle_g\big] = \langle u|v\rangle_{\mathring{L}^2_g} \qquad \forall u, v \in \mathring{H}_g^{n/2+\varepsilon} . \tag{64}$$

Vice versa, given any grounded white noise ξ on (M, g), then $h := \sqrt{\mathsf{k}_g}\, \xi$ is a co-polyharmonic Gaussian field on (M, g).

Remark 3.5 (cf. Proposition 3.9). Let (M, g) be an admissible manifold and h an $\mathring{H}_g^{-\varepsilon}$-valued co-polyharmonic Gaussian field on it, defined on some probability space $(\Omega, \mathfrak{F}, \mathbf{P})$.

Then, in analogy to the definition of the Itō integral and in view of (61), the mapping $\langle h|\cdot\rangle_g : \mathring{H}_g^{n/2} \to L^2(\mathbf{P})$ extends to a linear isometry

$$\langle\!\langle h|\cdot\rangle\!\rangle_g : \mathring{H}_g^{-n/2} \to L^2(\mathbf{P}) \tag{65}$$

in the spirit of Itō's L^2-isometry.

The heuristic characterization (60) of the measure CGF_g manifests itself in various important properties. As every Gaussian measure, CGF_g satisfies a *large deviation principle* whose rate function is given by the Cameron–Martin norm [4, Chap. II, Prop. 1.5 and Thm. 1.6]. In our case, this yields the following.

Proposition 3.6. *For every co-polyharmonic field h, and for every Borel set $A \subset \mathring{H}_g^{-\varepsilon}$:*

$$-\inf_{u\in A^0} \mathfrak{p}_g(u) \leqslant \liminf_{\beta\to 0} 2\beta^2\, \mathbf{P}[\beta h \in A]$$
$$\leqslant \limsup_{\beta\to 0} 2\beta^2\, \mathbf{P}[\beta h \in A] \leqslant -\inf_{u\in \bar{A}} \mathfrak{p}_g(u).$$

Here, A^0 and $\bar{A}$ respectively denote the interior and the closure of A in the topology of $\mathring{H}_g^{-\epsilon}$ for given $\epsilon > 0$, the functional $\mathfrak{p}_g$ is defined in (40) and we set $\mathfrak{p}_g(u) = \infty$ if $u \notin H_g^{n/2}$.

Next, we recall the celebrated *change of variable formula of Girsanov type*, also known as Cameron–Martin theorem, see, for instance, [52, Theorem 14.1].

Proposition 3.7. *If $\varphi \in \mathring{H}_g^{n/2}$ and $h \sim \mathsf{CGF}_g$, then $h + \varphi$ is distributed according to*

$$\exp\Big(\langle\!\langle h|\mathsf{p}_g\varphi\rangle\!\rangle_g - \frac{1}{2}\mathfrak{p}_g(\varphi,\varphi)\Big)\, d\mathsf{CGF}_g(h).$$

For $\varphi \in \mathring{H}_g^s$ with $s > n$, the $L^2(\mathbf{P})$-random variable $\langle\!\langle h|\mathsf{p}_g\varphi\rangle\!\rangle_g$ can be equivalently replaced by the usual pairing $\langle h|\mathsf{p}_g\varphi\rangle_g$.

Remark 3.8. Many of our subsequent results rely on the seminal work of J.-P. Kahane [54] on Gaussian multiplicative chaos. His results apply to Gaussian random fields $\tilde{h}$ on a metric space (M, d) with covariance kernel $\tilde{k}$ with a logarithmic divergence: $|\tilde{k}(x, y) + \log d(x, y)| \leqslant C$. In addition to nonnegative definiteness, he assumes that $\tilde{k}$ is nonnegative. Of course, this is not satisfied by our kernel k_g. However, as we are going to explain now, it imposes no serious obstacle to applying his results in our setting.

Given the kernel k_g as defined above, observe that it is smooth outside the diagonal and positive in the neighborhood of the diagonal. Define a new kernel by

$$\bar{k}(x, y) := k_g(x, y) + C \geqslant 0$$

with $C := -\min_{x,y\in M} k_g(x, y) < \infty$. By construction, $\bar{k}$ is nonnegative. Furthermore, it is also nonnegative definite since it is the covariance kernel for the Gaussian field

$$\bar{h} := h + \sqrt{C}\,\xi$$

where h denotes the co-polyharmonic Gaussian field associated with k_g, and ξ denotes a standard Gaussian variable independent of h.

3.2 | Approximations

As anticipated, our goal is to construct the random measure $d\mu(x) = e^{h(x)}\mathrm{dvol}_g(x)$. Due to the nonsmooth nature of h, this requires approximating h by smooth fields (and properly renormalizing). Co-polyharmonic Gaussian Fields may be approximated in various ways, the random measure obtained being essentially independent on the choice of the approximation [78]. Here, we present a number of different approximations: through their expansion in terms of eigenfunctions of the normalized co-polyharmonic operator p_g; by convolution with (smooth or nonsmooth) functions; by a discretization procedure.

Let us first discuss the eigenfunctions approximation. As before, we denote by $(\psi_j)_{j\in\mathbb{N}_0}$ the complete L^2-orthonormal system consisting of eigenfunctions of p_g, each with corresponding eigenvalue ν_j. In addition, we consider a sequence $(\xi_j)_{j\in\mathbb{N}_1}$ of independent and identically distributed standard Gaussian variables. For each $\ell \in \mathbb{N}_0$, we define the random test function

$$h_\ell(x) := \sum_{j=1}^{\ell} \frac{1}{\sqrt{\nu_j}} \psi_j(x)\,\xi_j, \qquad x \in M \,. \tag{66}$$

The covariance of the random field h_ℓ is given by:

$$k_\ell(x,y) := \mathbf{E}\big[h_\ell(x)\,h_\ell(y)\big] = \sum_{j=1}^{\ell} \frac{1}{\nu_j} \psi_j(x)\,\psi_j(y)\,, \qquad x,y \in M \,. \tag{67}$$

Our next result establishes that the random field h_ℓ converges to the random field h.

Proposition 3.9. *Let (M,g) be admissible and $\left(h_\ell\right)_{\ell\in\mathbb{N}}$ defined as above. Then*

(i) *For all $\varepsilon > 0$, the field h_ℓ, regarded as a random element of $\mathring{H}_g^{-\varepsilon}$, converges as $\ell \to \infty$ to a co-polyharmonic Gaussian field h in $L^2(\mathbf{P})$ and $\mathbf{P}$-a.s. In particular, $h \in \mathring{H}_g^{-\varepsilon}$ $\mathbf{P}$-a.s. Moreover, $h \notin L^2_g$ $\mathbf{P}$-a.s.*

(ii) *For every $u \in \mathring{H}_g^{-n/2}$, the sequence $\left(\langle u|h_\ell\rangle_g\right)_{\ell\in\mathbb{N}}$ is a centered, L^2-bounded martingale on $(\Omega, \mathfrak{F}, \mathbf{P})$ converging to $\langle\!\langle h|u\rangle\!\rangle_g$ $\mathbf{P}$-a.s. and in $L^2(\mathbf{P})$ as $\ell \to \infty$, cf. Remark 3.5. In this sense, $\mathbf{P}$-a.s.*

$$\langle\!\langle h|u\rangle\!\rangle_g = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\nu_j}} \langle u|\psi_j\rangle_g\, \xi_j \,.$$

Proof. The proof follows from the abstract construction of [44] and Definition 3.1. For completeness, we outline a simple proof in our setting.

(i) Let ℓ and $p \in \mathbb{N}$, and $\varepsilon > 0$. According to Lemma 2.15(ii), we have that, $\mathbf{P}$-almost surely

$$\left\| \sum_{j=\ell+1}^{p} \frac{\psi_j \xi_j}{\sqrt{\nu_j}} \right\|^2_{\mathring{H}^{-\varepsilon}} = \sum_{j=\ell+1}^{p} \frac{\xi_j^2}{\nu_j^{1+2\varepsilon/n}} \simeq \sum_{j=\ell+1}^{p} \frac{\xi_j^2}{j^{1+2\varepsilon/n}} \,.$$

The sum on the right-hand side is a generalized chi-square random variable with variance $\sum_{j=\ell+1}^{p} j^{-2\varepsilon/n-1}$. It converges as $p \to \infty$ if and only if $\varepsilon > 0$. This shows that the series

$$h := \sum_{j=1}^{\infty} \frac{\psi_j\, \xi_j}{\sqrt{\nu_j}} \ , \tag{68}$$

exists $\mathbf{P}$-almost surely in $\mathring{H}_g^{-\varepsilon}$ but $h \notin L_g^2$. The proof of the convergence in $L^2(\mathbf{P})$ is carried out in the same way. Since h is an $L^2(\mathbf{P})$-limit of Gaussian fields, it is itself Gaussian. For $u, v \in \mathring{H}_g^{n/2}$, its covariance is given by

$$\mathbf{E}\big[\langle h|u\rangle_g \langle h|v\rangle_g\big] = \sum_{j=1}^{\infty} \frac{1}{\nu_j} \langle \psi_j|u\rangle_{L_g^2} \langle \psi_j|v\rangle_{L_g^2} = \mathcal{K}(u,v) \ . \tag{69}$$

(ii) For all $u \in \mathring{H}_g^{-n/2}$, the sequence $\big(\langle u|h_\ell\rangle_g\big)_{\ell\in\mathbb{N}}$ is a martingale as a sum of independent and identically distributed random variables. Moreover, by orthogonality,

$$\sup_{\ell\in\mathbb{N}} \mathbf{E}\Big[\big\langle u \,\big|\, h_\ell \big\rangle_g^2\Big] = \sum_{j=1}^{\infty} \frac{1}{\nu_j} \langle u|\psi_j\rangle_g^{\ 2} = \mathcal{K}(u,u) = \|u\|^2_{\mathring{H}_g^{-n/2}} < \infty \ .$$

Thus, the martingale is $L^2(\mathbf{P})$-bounded and convergence follows from Doob's Martingale Convergence Theorem. The limit is the requested $\langle h|u\rangle_g \in L^2(\mathbf{P})$. □

The previous result allows us to construct a co-polyharmonic Gaussian field on every probability space that supports a sequence of independent and identically distributed standard normal variables. It is also important to know that an approximation $h_\ell \to h$ as in the previous proposition holds for every co-polyharmonic Gaussian field, independently of the construction of the latter.

Remark 3.10. Given any co-polyharmonic Gaussian field h, and the sequence of eigenfunctions $(\psi_j)_{j\in\mathbb{N}_0}$ as above, define a sequence $(\xi_j)_{j\in\mathbb{N}}$ of independent and identically distributed standard normal variables by setting $\xi_j := \big\langle h \,|\, \psi_j \big\rangle_g$ for all $j \in \mathbb{N}$, and a sequence of Gaussian random fields $(h_\ell)_{\ell\in\mathbb{N}}$ by

$$h_\ell : \Omega \longrightarrow \mathring{L}_g^2 \ , \qquad h_\ell^\omega(x) := \sum_{j=1}^{\ell} \frac{\psi_j(x)}{\sqrt{\nu_j}} \big\langle h^\omega \,\big|\, \psi_j \big\rangle_g \ . \tag{70}$$

Then, for every $u \in \mathring{H}_g^{n/2}$, as $\ell \to \infty$,

$$\big\langle h_\ell \,\big|\, u \big\rangle_g \longmapsto \big\langle h \,\big|\, u \big\rangle_g \qquad \mathbf{P}\text{-a.s. and in } L^2(\mathbf{P}) \ .$$

Now, let us consider more general approximations. The previous eigenfunction approximation will appear as a particular case.

Proposition 3.11. *Let h be a co-polyharmonic Gaussian field on (M,g), and for each $\ell\in\mathbb{N}$ let $q_\ell\in L^2(M^2,\mathsf{vol}_g\otimes\mathsf{vol}_g)$ be such that $\mathsf{q}_\ell u\to u$ in L^2_g for all $u\in\mathring{L}^2_g$, where*

$$\mathsf{q}_\ell u(x):=\left\langle q_\ell(x,\cdot\,)\,\middle|\,u\right\rangle_{L^2_g}\ .$$

(i) *Then, for every $\ell\in\mathbb{N}$, the field of functions h_ℓ on M defined by*

$$h_\ell(y)=(\mathsf{q}^*_\ell h)(y):=\langle\!\langle h|q_\ell(\,\cdot\,,y)\rangle\!\rangle_g \tag{71}$$

is a centered Gaussian field with covariance function

$$k_\ell(x,y)=\big((\mathsf{q}_\ell\otimes\mathsf{q}_\ell)K\big)(x,y):=\iint K(x',y')\,q_\ell(x,x')\,q_\ell(y,y')\,\mathrm{d}\mathsf{vol}_g(y')\,\mathrm{d}\mathsf{vol}_g(x')\ . \tag{72}$$

(ii) *As $\ell\to\infty$, for every $u\in\mathring{L}^2_g$,*

$$\left\langle h_\ell\,\middle|\,u\right\rangle_{L^2_g}\longmapsto\langle\!\langle h|u\rangle\!\rangle_g\qquad\mathbf{P}\text{-a.s. and in } L^2(\mathbf{P})\ . \tag{73}$$

Proof. (i) is obvious. To see (ii), observe that $\langle h_\ell\,|\,u\rangle_{L^2_g}=\langle h\,|\,\mathsf{q}_\ell u\rangle_g$, and thus, by (61) for every $u\in\mathring{L}^2_g$,

$$\begin{aligned}\mathbf{E}\left[\left|\langle\!\langle h|u\rangle\!\rangle_g-\left\langle h_\ell\,\middle|\,u\right\rangle_{L^2_g}\right|^2\right]&=\mathbf{E}\left[\left|\langle\!\langle h|u-\mathsf{q}_\ell u\rangle\!\rangle_g\right|^2\right]=\|u-\mathsf{q}_\ell u\|^2_{\mathring{H}^{-n/2}_g}\\&\leqslant C\,\|u-\mathsf{q}_\ell u\|^2_{\mathring{L}^2_g}\xrightarrow{\ell\to\infty}0\end{aligned}$$

where the last inequality follows from 2.15(v). □

Example 3.12.

(i) *Probability kernels.* Let $\{q_\ell(x,\cdot\,)\mathsf{vol}_g:\ell\in\mathbb{N},x\in M\}$ be a family of probability measures on M with $q_\ell\in L^\infty(\mathsf{vol}_g\otimes\mathsf{vol}_g)$ nonnegative, and such that $q_\ell(x,\cdot\,)\mathsf{vol}_g$ converges weakly to δ_x as $\ell\to\infty$ for each $x\in M$. Then $\mathsf{q}_\ell u\to u$ in L^2 as $\ell\to\infty$ for all $u\in L^2$.
Particular cases of (i) are (ii) and (iii) below.

(ii) *Discretization.* Let $(\mathfrak{P}_\ell)_{\ell\in\mathbb{N}}$ be a family of Borel partitions of M with

$$\sup\{\mathrm{diam}_g(A):A\in\mathfrak{P}_\ell\}\to 0\quad\text{as}\quad\ell\to\infty\ .$$

For $\ell\in\mathbb{N}$, put

$$q_\ell(x,y):=\sum_{A\in\mathfrak{P}_\ell}\frac{1}{\mathsf{vol}_g(A)}\mathbf{1}_A(x)\mathbf{1}_A(y)\ .$$

In other words, for given $x\in M$, we have that $q_\ell(x,\cdot\,)=\frac{1}{\mathsf{vol}_g(A)}\mathbf{1}_A$ with the unique $A\in\mathfrak{P}_\ell$ that contains x. Letting h_ℓ be defined as in Proposition 3.11 then yields

$$h_\ell(x)=\frac{1}{\mathsf{vol}_g(A)}\left\langle h\,\middle|\,\mathbf{1}_A\right\rangle_g\qquad A\in\mathfrak{P}_\ell\ ,\ x\in A\ .$$

This is a centered Gaussian random field $(h_\ell(x))_{x\in M}$ with covariance function

$$k_\ell(x,y) = \frac{1}{\mathrm{vol}_g(A^x_\ell)\,\mathrm{vol}_g(A^y_\ell)} \int_{A^x_\ell}\int_{A^y_\ell} k(x',y')\,d\mathrm{vol}_g(x')\,d\mathrm{vol}_g(y')\ ,$$

where A^x_ℓ is the unique element of $\mathfrak{P}_\ell$ containing x.

(iii) *Heat kernel approximation.* Let $q_\ell(x,y) := p_{1/\ell}(x,y)$ be defined in terms of the heat kernel on M. Then $\mathsf{q}_\ell u \to u$ in L^2 and thus in particular (73) holds for all $u \in L^2$. Even more, (73) holds for all $u \in \mathring{H}^{-n/2}$.

(iv) *Eigenfunctions approximation.* In terms of the eigenfunctions for the co-polyharmonic operator P_g, we define

$$q_\ell(x,y) := \sum_{j=0}^{\ell} \psi_j(x)\,\psi_j(y)\ .$$

In other words, $\mathsf{q}_\ell : L^2 \to L^2$ is the projection onto the linear span of the first $1+\ell$ eigenfunctions. Then, $\mathsf{q}_\ell u \to u$ in L^2 as $\ell\to\infty$ for all $u \in L^2$.

Proof.

(i) Since $C_b(M)$ is dense in $L^2(X)$ and since by Jensen's inequality $\|\mathsf{q}_\ell u - \mathsf{q}_\ell v\|_{L^2} \leqslant \|u-v\|_{L^2}$, it suffices to prove that $\mathsf{q}_\ell u \to u$ in L^2 as $\ell\to\infty$ for $u\in C_b(M)$. To see the latter, observe that $\mathsf{q}_\ell u(x) \to u(x)$ for each x by weak convergence of $q_\ell(x,\cdot)\mathrm{vol}_g$ to δ_x, and that $\|\mathsf{q}_\ell u\|_{L^\infty} \leqslant \|u\|_{L^\infty} < \infty$.

(ii) is straightforward.

(iii) If $q_\ell = p_{1/\ell}$ and $u \in \mathring{H}^{-n/2}$, we have with $v := \mathring{\mathsf{G}}^{n/4}u \in L^2$,

$$\|u - \mathsf{q}_\ell u\|_{\mathring{H}^{-n/2}} \lesssim \|v - \mathsf{q}_\ell v\|_{L^2} \xrightarrow{\ \ell\to\infty\ } 0,$$

where the last inequality follows from 2.15(v).

(iv) Readily follows from the fact that $(\psi_j)_{j\in\mathbb{N}_0}$ is a complete L^2-orthonormal system, Lemma 2.15(iii). □

3.3 | Conformal quasi-invariance

Theorem 3.13. *Consider an admissible Riemannian manifold (M,g) and $g' = e^{2\varphi}g$ with $\varphi\in C^\infty(M)$. If h is distributed according to CGF_g, then*

$$h' := h - \langle h\rangle_{g'} \tag{74}$$

is distributed according to $\mathsf{CGF}_{g'}$.

Proof. By construction, h is a centered Gaussian random field on $\mathring{H}^{-\varepsilon}_g$ for some/any $\varepsilon>0$. Let us choose $\varepsilon = n/2$. According to Proposition 2.22, the random field h' as defined above then is a centered Gaussian random field on $\mathring{H}^{-n/2}_{g'}$. Moreover, for all $u,v \in \mathring{H}^{-n/2}_{g'}$ (which implies

$e^{n\varphi}u, e^{n\varphi}v \in \mathring{H}_g^{-n/2}$),

$$\begin{aligned}\mathbf{E}\Big[\big\langle h' \,|\, u\big\rangle_{g'} \cdot \big\langle h' \,|\, v\big\rangle_{g'}\Big] &\overset{(45)}{=} \mathbf{E}\Big[\big\langle h' \,|\, e^{n\varphi}u\big\rangle_{g} \cdot \big\langle h' \,|\, e^{n\varphi}v\big\rangle_{g}\Big] \\ &\overset{(74)}{=} \mathbf{E}\Big[\big\langle h \,|\, e^{n\varphi}u\big\rangle_{g} \cdot \big\langle h \,|\, e^{n\varphi}v\big\rangle_{g}\Big] \\ &\overset{(61)}{=} \mathcal{K}_g(e^{n\varphi}u, e^{n\varphi}v) \overset{(58)}{=} \mathcal{K}_{g'}(u,v).\end{aligned}$$

Thus, h' shares the defining properties of the co-polyharmonic field on (M, g'). □

The conformal quasi-invariance of the $\mathsf{CGF}_{M,g}$ indeed holds true in a more general form.

Theorem 3.14. *Assume that (M, g) and (M', g') are conformally equivalent with diffeomorphism Φ and conformal weight $e^{2\varphi}$ such that $\Phi^* g' = e^{2\varphi} g$. Furthermore, assume that h is distributed according to $\mathsf{CGF}_{M,g}$ and h' is distributed according to $\mathsf{CGF}_{M',g'}$. Then*

$$h' \overset{(\mathrm{d})}{=} \big(h - \langle h\rangle_{g'}\big)\circ\Phi^{-1} \,. \tag{75}$$

Proof. Assume that h is distributed according to $\mathsf{CGF}_{M,g}$. Then with $g^* := e^{2\varphi} g$ by the previous Theorem 3.13, the field

$$h^* := h - \langle h\rangle_{g^*}$$

is distributed according to CGF_{M,g^*}. Thus, for the proof of the claim, we may assume without restriction that $\varphi = 0$ and $g^* = g, h^* = h$. In other words, assume that (M, g) and (M', g') are *isometric* with diffeomorphism $\Phi : M \to M'$ satisfying $\Phi^* g' = g$ ("pull back of the metrics"). Then $\Phi_* \mathsf{vol}_{M,g} = \mathsf{vol}_{M',g'}$ ("push forward of the measurers") and, by the uniqueness of the co-polyharmonic operator as stated in Theorem 1.3(vi),

$$\mathsf{P}_{M',g'} u = \mathsf{P}_{M,g}(u\circ\Phi)\circ\Phi^{-1}.$$

This invariance of the co-polyharmonic operators carries over to the kernels of their normalized inverse

$$k_{M',g'}\big(x', y'\big) = k_{M,g}\big(\Phi^{-1}(x'), \Phi^{-1}(y')\big) \qquad \forall x', y' \in M'$$

as well as to the associated bilinear forms

$$\mathcal{K}_{M',g'}\big(u', v'\big) = \mathcal{K}_{M,g}\big(u'\circ\Phi, v'\circ\Phi\big) \qquad \forall u, v \in H^{-n/2}_{M',g'}.$$

Consequently, if h is distributed according to $\mathsf{CGF}_{M,g}$, then $h' := h\circ\Phi^{-1}$ is distributed according to $\mathsf{CGF}_{M',g'}$. □

To get rid of the additive correction term in (74), one can consider the "random variable" $h + a$, called *ungrounded co-polyharmonic Gaussian field*, where h is distributed according to $\mathsf{CGF}^{M,g}$ and where a is a constant distributed according to the Lebesgue measure on the line (the latter not being a probability measure).

More formally, given any admissible manifold (M, g), the distribution of the corresponding co-polyharmonic Gaussian field is a probability measure $\boldsymbol{\nu}_g$ on the grounded Sobolev space $\mathring{H}_g^{-\varepsilon}$. To override the influence of additive constants, we consider the (nonfinite) measure $\widehat{\boldsymbol{\nu}}_g$ on the (ungrounded) Sobolev space $H_g^{-\varepsilon}$ defined as the image measure of $\boldsymbol{\nu}_g \otimes \mathfrak{L}^1$ under the map

$$(h, a) \mapsto h + a \ .$$

Definition 3.15. The measure $\widehat{\boldsymbol{\nu}}_g$ is called *law of the ungrounded co-polyharmonic Gaussian field* and denoted by $\widehat{\mathsf{CGF}}_g$.

We write $\hbar \sim \widehat{\mathsf{CGF}}_g$ to indicate that a measurable map $\hbar : \Omega \to H_g^{-\varepsilon}$, defined on some measure space $(\Omega, \mathfrak{F}, \mathsf{m})$, is distributed according to $\widehat{\mathsf{CGF}}_g$, that is, $\hbar_* \mathsf{m} = \widehat{\mathsf{CGF}}_g$.

The conformal quasi-invariance of the probability measures CGF_g leads to a conformal invariance of the measures $\widehat{\mathsf{CGF}}_g$.

Proposition 3.16. *If $\hbar \sim \widehat{\mathsf{CGF}}_g$ and $\hbar' \sim \widehat{\mathsf{CGF}}_{g'}$ with $g' = e^{2\varphi} g$, then*

$$\hbar' \stackrel{(\mathrm{d})}{=} \hbar \ .$$

Proof. Let h be a (grounded) co-polyharmonic field on (M, g) and $h' := h - \langle h \rangle_{g'}$. Then, for all $F \in L^1(H_{g'}^{-\varepsilon}, \widehat{\boldsymbol{\nu}}_{g'})$, by translation invariance of one-dimensional Lebesgue measure and Theorem 3.13,

$$\begin{aligned} \int F \, d\widehat{\boldsymbol{\nu}}_{g'} &= \int_{\mathbb{R}} \mathbf{E}'[F(h' + a)] \, da = \int_{\mathbb{R}} \mathbf{E}[F(h - \langle h \rangle_{g'} + a)] \, da \\ &= \int_{\mathbb{R}} \mathbf{E}[F(h + a)] \, da = \int F \, d\widehat{\boldsymbol{\nu}}_g \ . \end{aligned}$$

□

Corollary 3.17. *On each class $(M, [g])$ of conformally equivalent admissible Riemannian manifolds, $\widehat{\mathsf{CGF}}_{M,g}$ defines a conformally invariant random field.*

The change of variable formula of Girsanov type of Proposition 3.7 extends from shifts $\varphi \in \mathring{H}_g^{n/2}$ to shifts $\varphi \in H_g^{n/2}$.

Corollary 3.18. *If $\varphi \in H_g^{n/2}$ and $\hbar \sim \widehat{\mathsf{CGF}}_g$, then $\hbar + \varphi$ is distributed according to*

$$\exp\left(\langle\!\langle \hbar | \mathsf{p}_g \varphi \rangle\!\rangle_g - \frac{1}{2} \mathfrak{p}_g(\varphi, \varphi) \right) d\widehat{\mathsf{CGF}}_g(\hbar) \, .$$

4 | THE LIOUVILLE QUANTUM GRAVITY MEASURE

Fix an admissible manifold (M, g) and a co-polyharmonic Gaussian field $h : \Omega \to \mathring{H}_g^{-\varepsilon}$, our naive goal is to study the "random geometry" on M obtained by the random conformal transformation,

$$e^{2h} g \ ,$$

and in particular to study the associated "random volume measure" given as

$$e^{nh(x)}\mathrm{dvol}_g(x)\ . \tag{76}$$

It easily can be seen that — due to the singular nature of the noise h — all approximating sequences of this measure diverge as long as no additional renormalization is built in.

A more tractable goal is to study (for suitable $\gamma \in \mathbb{R}$) the random measure $\mu^{\gamma h}$ formally given as

$$d\mu^{\gamma h}(x) = e^{\gamma h(x) - \frac{\gamma^2}{2}\mathbf{E}[h(x)^2]}\mathrm{dvol}_g(x). \tag{77}$$

Since h is not a function but only a distribution, both (76) and (77) are ill-defined. However, replacing h by its finite-dimensional noise approximation h_ℓ as constructed in Proposition 3.10 leads to a sequence $(\mu^{\gamma h})_\ell$ of random measures on M that, as $\ell \to \infty$, almost surely, converges to a random measure $\mu^{\gamma h}$ on M, the *plain LQG measure* on the n-dimensional manifold M. Let $\mathcal{M}_b(M)$ denote the set of finite positive Borel measures on M. We equip it with the Borel σ-algebra associated with its usual *weak topology*.

4.1 | Gaussian multiplicative chaos

In the following theorem, we construct the Gaussian multiplicative chaos $\mu^{\gamma h}$ associated to a co-polyharmonic Gaussian field h on (M, g). In view of Theorem 3.9, we can look at the co-polyharmonic Gaussian field h as a random element of $H_g^{-\varepsilon}$ for any $\varepsilon > 0$.

Theorem 4.1. *Let an admissible manifold (M, g) and a real number γ with $|\gamma| < \sqrt{2n}$ be given as well as a $\mathring{H}_g^{-\varepsilon}$ co-polyhmarmonic Gaussian field h, defined on some probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Then, there exists a measurable map*

$$\mu_g^{\gamma\bullet} : \mathring{H}_g^{-\varepsilon} \to \mathcal{M}_b(M), \qquad f \mapsto \mu_g^{\gamma f}\ , \tag{78}$$

with the following properties:

(i) *for $\mathbf{P}$-a.e. h and every $\varphi \in \mathring{H}_g^{n/2}$,*

$$\mu_g^{\gamma(h+\varphi)} = e^{\gamma\varphi}\,\mu_g^{\gamma h}, \tag{79}$$

(ii) *for all Borel measurable $f : \mathring{H}_g^{-\varepsilon} \times M \to [0, \infty]$, we have that*

$$\mathbf{E}\int f(h,x)d\mu_g^{\gamma h}(x) = \mathbf{E}\int f\big(h + \gamma k_g(x,\cdot), x\big)\,\mathrm{dvol}_g(x), \tag{80}$$

(iii) *for all $p \in (-\infty, \frac{2n}{\gamma^2})$,*

$$\mathbf{E}\Big[\mu_g^{\gamma h}(M)^p\Big] < \infty\ .$$

Remark 4.2. Equation (80) implies that $\mathbf{E}[\mu_g^{\gamma h}] = \mathrm{vol}_g$.

Definition 4.3. The random measure $\mu_g^{\gamma h}$ is called the *plain LQG measure on* (M,g).

Proof. The result follows from general results regarding the theory of Gaussian multiplicative chaos by Kahane [54] and Shamov [78]. Shamov [78] gives an axiomatic definition of Gaussian multiplicative chaos and shows that the limit measure is, in fact, independent of the choice of approximating sequence. In the language of [78], our result follows from the existence of a *subcritical Gaussian multiplicative chaos over the Gaussian field* h, identified with the mapping $\langle h|\cdot\rangle_g : \mathring{H}_g^{-n/2} \to L^2(\boldsymbol{\nu}_g)$, and the operator $\mathsf{k}_g : \mathring{H}_g^{-n/2} \to \mathring{H}_g^{n/2} \subset L_g^0$. Properties (i) and (ii) being respectively [78, Dfn. 11 (3)] and [78, Thm. 4]. The moments estimates (iii) can be found in [54, Thm. 4] for $p > 0$ and [75, Thm. 2.12] for $p < 0$.

The existence of the Gaussian multiplicative chaos $\mu_g^{\gamma h}$ — or, more precisely, the existence of the random variables $\int_M u\, d\mu_g^{\gamma h}$ as a limit of uniformly integrable martingales — follows from an argument of [54, Thm. 4, Variant 1]. This argument is stated in the slightly more restrictive setting of positive kernels. This restriction, however, does not harm in our case. Indeed, passing from h to $\hat{h} := h + C\xi$ with some standard normal variable ξ independent of h will change k_g into $\hat{k}_g + C^2$ that is eventually (for sufficiently large C) a positive kernel. The corresponding random measures are then related to each other according to

$$\hat{\mu}_g^{\gamma h} = \exp\left(\gamma C\xi - \tfrac{\gamma^2}{2}C^2\right)\mu_g^{\gamma h} . \qquad \square$$

Remark 4.4. Regarding uniform integrability and the existence of (plain) LQG measure, the work [7] provides an alternative approach based on the study of thick points of the underlying Gaussian fields.

4.2 | Approximations

Let us recall the content of [78, Thm. 25] specified to our setting.

Lemma 4.5. *Let $q_\ell \in L^2(M^2, \mathrm{vol}_g \otimes \mathrm{vol}_g)$ be a family of kernels as in Proposition 3.11 and let $(h_\ell(y))_{y\in M}$ be Gaussian fields as in (71) with covariance kernel $k_{g,\ell}$ as in (72). Further set*

$$d\mu_g^{\gamma h_\ell}(x) := \exp\left(\gamma h_\ell(x) - \frac{\gamma^2}{2}k_{g,\ell}(x,x)\right) d\mathrm{vol}_g(x) . \tag{81}$$

Assume that

(i) *The family $(\mu^{h_\ell}(M))_{\ell\in\mathbb{N}}$ is uniformly integrable.*
(ii) *For all $u \in \mathring{H}_g^{n/2}$, $\mathsf{q}_\ell u \to u$ in $L^0(M, \mathrm{vol}_g)$.*
(iii) *$k_{g,\ell} \to k_g$ in $L^0(M^2, \mathrm{vol}_g \otimes \mathrm{vol}_g)$.*

Then $\mu_g^{\gamma h_\ell} \to \mu_g^{\gamma h}$ weakly as Borel measures on M in **P***-probability as $\ell \to \infty$. Even more, for every $u \in L^1(M, \mathrm{vol}_g)$,*

$$\int_M u\, d\mu_g^{\gamma h_\ell} \to \int_M u\, d\mu_g^{\gamma h} \quad \text{in } L^1(\mathbf{P}) \quad \text{as} \quad \ell \to \infty . \tag{82}$$

In practice, the criteria (ii) and (iii) of the previous lemma are easy to verify. The remaining challenge is the verification of (i).

Lemma 4.6. *Assume that for every $\vartheta > 1$, there exists $C \geqslant 0$, $\ell_\vartheta \in \mathbb{N}$, and a nondecreasing sequence $(c_\ell)_{\ell\in\mathbb{N}}$ such that for all $\ell \geqslant \ell_\vartheta$ and all $x, y \in M$,*

$$k_{g,\ell}(x,y) \leqslant \vartheta \log\left(\frac{1}{d(x,y)} \wedge c_\ell\right) + C \ . \tag{83}$$

Then for every $\gamma \in (0, \sqrt{2n})$, the family $(\mu_g^{\gamma h_\ell}(M))_{\ell\in\mathbb{N}}$ as in (81) is uniformly integrable.

Proof. This follows from Kahane's comparison lemma [54], cf. [78, Theorems 27, 28]. □

In the rest of this section, let q_ℓ be a family of probability kernels as in Example 3.12 (i). The lemmas above allow us to obtain the following crucial approximation results.

Theorem 4.7. *Let the kernels q_ℓ be given in terms of a compactly supported, nonincreasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ as*

$$q_\ell(x,y) := \frac{1}{N_\ell(x)}\eta(\ell\, d(x,y)) \ , \qquad N_\ell(x) := \int_M \eta(\ell\, d(x,y))\, \mathrm{dvol}_g(y) \ .$$

Then, with $\mu_g^{\gamma h_\ell}$ defined as in (81),

$$\mu_g^{\gamma h_\ell} \to \mu_g^{\gamma h} \quad \textit{as} \quad \ell \to \infty$$

in the sense made precise in (82).

Remark 4.8. The assertion of the previous theorem holds as well for

$$q_\ell(x,y) := \frac{1}{N_\ell^*}\eta(\ell\, d(x,y))$$

with the "Euclidean normalization" $N_\ell^* := \ell^{-n} \int_{\mathbb{R}^n} \eta(|y|)\, d\mathcal{L}^n(y)$ in the place of the "Riemannian normalization" $N_\ell(x)$.

Proof. Assume that η is supported in $[0, R]$. The verification of the criteria (ii) and (iii) in Lemma 4.5 is straightforward: (ii) was proven in Example 3.12 (i). (iii) follows from the fact that $k_g(x,\cdot)$ is bounded and continuous outside of any ε-neighborhood of x, that $q_\ell(x,\cdot)$ is supported in an R/ℓ-neighborhood of y, and that $q_\ell(x,\cdot) \to \delta_x$ as $\ell \to \infty$. Thus, for every x, y with $d(x,y) \geqslant 2\varepsilon$ and every $\ell \geqslant R/\varepsilon$,

$$k_{g,\ell}(x,y) = \int_{B_\varepsilon(y)} \int_{B_\varepsilon(x)} k_g(x',y')\, q_\ell(x',x)\, q_\ell(y',y) \mathrm{dvol}_g(x')\, \mathrm{dvol}_g(y') \xrightarrow{\ell\to\infty} k_g(x,y) \ .$$

Our verification of the criterion (i) in Lemma 4.5 is based on Lemma 4.6, the verification of which will, in turn, be based on the following auxiliary results.

Claim 4.9. For all $x, y \in M$ with $d(x,y) \geqslant 3R/\ell$,

$$\iint \log \frac{1}{d(x',y')}\, q_\ell(x,x')\, q_\ell(y,y')\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \leqslant \log \frac{1}{d(x,y)} + \log 3 \,.$$

Proof. Combining the assumption $d(x,y) \geqslant 3R/\ell$ and the facts that $d(x,x') \leqslant R/\ell$ for all x' in the support of $q_\ell(x,\,\cdot\,)$ and $d(y,y') \leqslant R/\ell$ for all y' in the support of $q_\ell(y,\,\cdot\,)$ yields

$$d(x',y') \geqslant d(x,y) - d(x,x') - d(y,y') \geqslant d(x,y) - 2R/\ell \geqslant \frac{1}{3} d(x,y) \,.$$

Thus, the claim readily follows. □

Claim 4.10. For every $\vartheta > 1$, there exist $\ell_\vartheta \in \mathbb{N}$ such that

$$\int \log \frac{1}{d(x,z)}\, q_\ell(y,z)\, d\mathrm{vol}_g(z) \leqslant \frac{\vartheta}{N^*_\ell} \int_{\mathbb{R}^n} \log \frac{1}{|x'-z|}\, \eta\big(\ell\,|y'-z|\big)\, d\mathcal{L}^n(z) + \vartheta \log \vartheta$$

for all $\ell \geqslant \ell_\vartheta$, all $x, y \in M$ with $d(x,y) < 4R/\ell$, and all $x', y' \in \mathbb{R}^n$ with $d(x,y) = |x'-y'|$.

Proof. Denote by $\mathrm{inj}_g(M) > 0$ the injectivity radius of (M,g). For every $y \in M$, set $y' := 0 \in \mathbb{R}^n$ and use the exponential map $\exp_y : \mathbb{R}^n \to M$ to identify ε-neighborhoods of $y \in M$ with ε-neighborhoods of $y' \in \mathbb{R}^n$ for all $\varepsilon \in (0, \mathrm{inj}_g(M))$. Since M is compact and smooth, for every $\vartheta > 1$, there exists $\varepsilon_\vartheta \in (0, \mathrm{inj}_g(M))$ so small that $\exp_y$ deforms both distances and volume elements in ε-neighborhoods of y by a factor less than ϑ, for every $y \in M$ and every $\varepsilon \in (0, \varepsilon_\vartheta)$. Choose ℓ_ϑ so that $4R/\ell_\vartheta < \varepsilon_\vartheta$. Thus,

$$\begin{aligned}\int \log \frac{1}{d(x,z)} q_\ell(y,z)\, d\mathrm{vol}_g(z) &\leqslant \frac{\vartheta}{N^*_\ell} \int_{\mathbb{R}^n} \log \frac{\vartheta}{|x'-z|} \eta\big(\ell\,|y'-z|\big)\, d\mathcal{L}^n(z) \\ &= \frac{\vartheta}{N^*_\ell} \int_{\mathbb{R}^n} \log \frac{1}{|x'-z|} \eta\big(\ell\,|y'-z|\big)\, d\mathcal{L}^n(z) + \vartheta \log \vartheta \,.\end{aligned}$$

□

Claim 4.11. For all $x, y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \log \frac{1}{|x-z|}\, \eta(\ell\,|y-z|)\, d\mathcal{L}^n(z) \leqslant \int_{\mathbb{R}^n} \log \frac{1}{|z|}\, \eta(\ell\,|z|)\, d\mathcal{L}^n(z) \,.$$

Proof. Without restriction $y = 0$ and $\ell = 1$. For $r \geqslant 0$, consider

$$\phi(r) := \int_{\mathbb{R}^n} \log \frac{1}{|rx-z|}\, \eta(|z|)\, d\mathcal{L}^n(z) \,.$$

Then

$$\begin{aligned}\phi'(r) &= \int_{\mathbb{R}^n} \frac{\langle rx-z, x\rangle}{|rx-z|^2}\, \eta(|z|)\, d\mathcal{L}^n(z) = \int_{\mathbb{R}^n} \frac{\langle z, x\rangle}{|z|^2}\, \eta(|z-rx|)\, d\mathcal{L}^n(z) \\ &= \int_{\{z:\langle z,x\rangle \geqslant 0\}} \frac{\langle z, x\rangle}{|z|^2}\, (\eta(|z-rx|) - \eta(|z+rx|))\, d\mathcal{L}^n(z) \leqslant 0\end{aligned}$$

since $t \mapsto \eta(t)$ is nonincreasing. □

Claim 4.12. There exists $C^* \geqslant 0$ such that, for all $\ell \in \mathbb{N}$,

$$\frac{1}{N^*_\ell} \int_{\mathbb{R}^n} \log \frac{1}{|z|}\, \eta(\ell\, |z|)\, d\mathcal{L}^n(z) \leqslant \log \ell + C^* \ .$$

Proof. Straightforward with $C^* := \frac{1}{N^*_1} \int_{\mathbb{R}^n} \log \frac{1}{|z|}\, \eta(|z|)\, d\mathcal{L}^n(z)$. □

Now let us conclude the *proof of Theorem* 4.7. Fix $\vartheta > 1$, and choose ℓ_ϑ as in the proof of Claim 4.10. It remains to verify the estimate (83). For x, y with $d(x,y) \geqslant 3R/\ell$, this is derived in Claim 4.9. For x, y with $d(x,y) < 3R/\ell$ (hence $d(x,y) < \varepsilon_\vartheta$), Claims 4.10–4.12 yield

$$\int \log \frac{1}{d(x',y')}\, q_\ell(y,y')\, d\mathrm{vol}_g(y') \leqslant \vartheta(\log \ell + C^*) + \vartheta \log \vartheta$$

for every x' in the support of $q_\ell(x,\,\cdot\,)$, and thus,

$$\iint \log \frac{1}{d(x',y')}\, q_\ell(x,x')\, q_\ell(y,y')\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \leqslant \vartheta(\log \ell + C^*) + \vartheta \log \vartheta \ .$$

This proves the estimate (83) with $c_\ell := \ell$ and $C := C^* \vartheta^2 \log \vartheta$, and the proof of the theorem is herewith complete. □

The previous results, in particular, apply to the kernel $q_\ell(x,y) := \frac{1}{\mathrm{vol}_g(B_{1/\ell}(x))} \mathbf{1}_{B_{1/\ell}(x)}(y)$. Similar arguments apply to discretization kernels.

Theorem 4.13. *Let $(\mathfrak{P}_\ell)_{\ell\in\mathbb{N}}$ be a family of partitions of M with $d_\ell := \sup\{\mathrm{diam}(A) : A \in \mathfrak{P}_\ell\} \to 0$ as $\ell \to \infty$, see Example 3.12(ii), and $\inf\{\mathrm{vol}_g(A)/d_\ell^n : A \in \mathfrak{P}_\ell, \ell \in \mathbb{N}\} > 0$. Let*

$$q_\ell := \sum_{A \in \mathfrak{P}_\ell} \frac{1}{\mathrm{vol}_g(A)} \mathbf{1}_A \otimes \mathbf{1}_A \ .$$

Then, with $\mu_g^{\gamma h_\ell}$ defined as in (81),

$$\mu_g^{\gamma h_\ell} \to \mu_g^{\gamma h} \quad \text{as } \ell \to \infty$$

in the sense made precise in (82).

Proof. Again, the argumentation will be based on Lemma 4.5. The verification of the criteria (ii) and (iii) there is again straightforward. Criterion (i) will be verified as before by means of Lemma 4.6. To verify (83), assume without restriction that $(\mathfrak{P}_\ell)_{\ell\in\mathbb{N}}$ is given with

$$\mathrm{diam}(A) \leqslant d_\ell, \qquad \mathrm{vol}_g(A) \geqslant v_\ell \geqslant V d_\ell^n$$

for all $A \in \mathfrak{P}_\ell$, $\ell \in \mathbb{N}$ and for some constant $V > 0$ independent of ℓ. Then, for $x, y \in M$ with $d(x,y) > 3d_\ell$, we obtain as in Claim 4.9 that

$$\iint \log \frac{1}{d(x',y')}\, q_\ell(x,x')\, q_\ell(y,y')\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \leqslant \log \frac{1}{d(x,y)} + \log 3 \ .$$

Furthermore, for every $\vartheta > 1$, every sufficiently large ℓ, and for all $x, y \in M$ with $d(x,y) \leqslant 3d_\ell$, we have that

$$\begin{aligned}
&\iint \log\frac{1}{d(x',y')}\, q_\ell(x,x')\, q_\ell(y,y')\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \\
&\quad\leqslant \sup_{x'\in A_x} \frac{1}{\mathrm{vol}_g(A_y)} \int_{A_y} \log\frac{1}{d(x',y')}\, d\mathrm{vol}_g(y') \\
&\quad\leqslant \sup_{x'\in\mathbb{R}^n} \sup_{\substack{A\subset\mathbb{R}^n\\ \mathcal{L}^n(A)\leqslant v_\ell}} \frac{\vartheta}{\mathcal{L}^n(A)} \int_A \log\frac{1}{|x'-y'|}\, d\mathcal{L}^n(y')
\end{aligned}$$

by comparison of Riemannian and Euclidean distances and volumes. Since $|x'-y'|$ is translation invariant, we may dispense with the supremum over x' and assume instead that $x' = 0 \in \mathbb{R}^n$. Furthermore,

$$\begin{aligned}
\sup_{\substack{A\subset\mathbb{R}^n\\ \mathcal{L}^n(A)\leqslant v_\ell}} \frac{1}{\mathcal{L}^n(A)} \int_A \log\frac{1}{|x'-y'|}\, d\mathcal{L}^n(y') &= \sup_{v\leqslant v_\ell} \sup_{\substack{A\subset\mathbb{R}^n\\ \mathcal{L}^n(A)=v}} \frac{1}{v}\int_{\mathbb{R}^n} \mathbf{1}_A(y') \log\frac{1}{|y'|}\, d\mathcal{L}^n(y') \\
&\leqslant \sup_{v\leqslant v_\ell} \frac{1}{v}\int_{B_r(0)} \log\frac{1}{|y'|}\, d\mathcal{L}^n(y')
\end{aligned}$$

by Hardy–Littlewood inequality and spherical symmetry of $-\log|y'|$, where $r = r(v)$ is so that $\mathcal{L}^n(B_r(0)) = v$. Furthermore, since $\ell \mapsto v_\ell$ is monotone decreasing to 0, we may choose ℓ additionally so large that $v_\ell \leqslant 1$. For all such ℓ, since $r(v) \leqslant r(v_\ell) \leqslant 1$ and $-\log|y'| \geqslant 1$ on $B_r(0)$, the function $v \mapsto \frac{1}{v}\int_{B_{r(v)}(0)} \log\frac{1}{|y'|}\, d\mathcal{L}^n(y')$ is increasing for $v \in [0, v_\ell)$. We have therefore that, for every $\vartheta > 1$, every sufficiently large ℓ, every $x, y \in M$ with $d(x,y) \leqslant 3d_\ell$,

$$\iint \log\frac{1}{d(x',y')}\, q_\ell(x,x')\, q_\ell(y,y')\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \leqslant \frac{1}{v_\ell}\int_{B_r(0)} \log\frac{1}{|y'|}\, d\mathcal{L}^n(y')$$

with $r > 0$ such that $\mathcal{L}^n(B_r(0)) = v_\ell$. For such r, we may compute

$$\begin{aligned}
\frac{1}{\mathcal{L}^n(B_r(0))}\int_{B_r(0)} \log\frac{1}{|y'|}\, d\mathcal{L}^n(y') &= \frac{n}{r^n}\int_0^r \log\frac{1}{s}\, s^{n-1} ds = \frac{1}{n\,r^n}\int_0^{r^n} \log\frac{1}{t}\, dt \\
&= \frac{1}{n\,r^n} r^n (1 - \log r^n) = \frac{1}{n} + \log\frac{1}{r}\ .
\end{aligned}$$

That is, for $d(x,y) \leqslant 3d_\ell$ with sufficiently large ℓ,

$$\iint \log\frac{1}{d(x',y')}\, q_\ell(x,x')\, q_\ell(y,y')\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \leqslant \vartheta\left(\frac{1}{n} + \log\frac{1}{r}\right) \leqslant C + \vartheta \log\frac{1}{3d_\ell}$$

since $r = \left(v_\ell/c_n\right)^{1/n} \geqslant (V/c_n)^{1/n}\, d_\ell$ with $c_n = \mathcal{L}^n(B_1(0))$. Thus, summarizing, for all $x, y \in M$ and all sufficiently large ℓ,

$$\iint \log\frac{1}{d(x',y')}\, q_\ell(x,x')\, q_\ell(y,y')\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \leqslant C + \vartheta \log\frac{1}{d(x,y) \vee 3d_\ell}\ . \qquad \square$$

The previous theorems do *not* apply to the kernels

$$q_\ell := \sum_{j=1}^{\ell} \psi_j \otimes \psi_j$$

for the eigenspace projections. These kernels are not nonnegative and not supported on small balls, even for large ℓ. Nevertheless, $\mu^{\gamma h}$ can also be obtained via eigenfunctions approximation according to our next result.

Theorem 4.14. *Consider the eigenfunctions approximation $\left(h_\ell\right)_{\ell\in\mathbb{N}}$ given in (66) with covariance kernel $\left(k_{g,\ell}\right)_{\ell\in\mathbb{N}}$ as in (67). Let $\left(\mu_g^{\gamma h_\ell}\right)_{\ell\in\mathbb{N}}$ be as in (81). Then for all Borel $B \subset M$,*

$$\mathbf{E}\left[\mu_g^{\gamma h}(B) \,|\, \xi_1, \dots, \xi_\ell\right] = \mu_g^{\gamma h_\ell}(B) \ .$$

In particular, $\left(\mu_g^{\gamma h_\ell}(B)\right)_{\ell\in\mathbb{N}}$ is a uniformly integrable martingale and thus $\mathbf{P}$-a.s. and in $L^1(\mathbf{P})$

$$\mu_g^{\gamma h_\ell}(B) \to \mu_g^{\gamma h}(B) \qquad \text{as } \ell \to \infty \ . \tag{84}$$

Proof. Fix $\varepsilon > 0$ and consider the function $F:\ C^\infty(M) \times \mathring{H}_g^{-\varepsilon}$

$$F(\Phi, \Psi) := \int_B e^{\gamma\Phi(x)} d\mu_g^{\gamma\Psi}(x), \qquad \Phi \in C^\infty(M), \Psi \in \mathring{H}_g^{-\varepsilon} \ .$$

Fix a Borel set $B \subset M$. Since h_ℓ is almost surely smooth, in view of (79), we find that

$$\mu_g^{\gamma h}(B) = F(h_\ell, h - h_\ell).$$

Again, by (79), $\mu_g^{\gamma(h-h_\ell)}(dx) = e^{-\gamma h_\ell(x)} \mu_g^{\gamma h}(dx)$, and we see that, for $\Phi \in C^\infty$ deterministic

$$G(\Phi) := \mathbf{E}F(\Phi, h - h_\ell) = \mathbf{E} \int_B e^{\gamma\Phi(x)} e^{-\gamma h_\ell(x)} d\mu_g^{\gamma h}(x) \ .$$

Observing that $h_\ell(x) = \pi_{\ell,x}(h)$ and

$$k_{g,\ell}(x, y) = \pi_{\ell,x}\big(k_g(x, \cdot)\big)(y) , \qquad \pi_{\ell,x}(u) := \sum_{j=1}^{\ell} \langle u|\psi_j\rangle_g \psi_j(x) , \quad u \in H_g^{-\varepsilon} ,$$

putting $f(h, x) := e^{\gamma\Phi(x)} e^{-\gamma\pi_{\ell,x}(h)}$, and applying (80), we find that

$$\begin{aligned} G(\Phi) &= \mathbf{E} \int_B f(h, x) d\mu_g^{\gamma h}(x) = \mathbf{E} \int_B f(h + \gamma k_g(x, \cdot), x) \mathrm{dvol}_g(x) \\ &= \mathbf{E} \int_B e^{\gamma\Phi(x)} e^{-\gamma h_\ell(x) - \gamma^2 k_{g,\ell}(x,x)} \mathrm{dvol}_g(x). \end{aligned}$$

Using that $h_\ell(x)$ is Gaussian, we conclude that

$$G(\Phi) = \int_B e^{\gamma\Phi(x)} e^{-\frac{\gamma^2}{2} k_{g,\ell}(x,x)} \mathrm{dvol}_g(x) \ .$$

Since h_ℓ and $h - h_\ell$ are independent and h_ℓ is measurable with respect to $\xi_1, \dots, \xi_\ell$, we have that

$$\mathbf{E}\Big[\mu_g^{\gamma h}(B)\,\Big|\,\xi_1,\dots,\xi_\ell\Big] = \mathbf{E}\big[F(h_\ell, h-h_\ell)\,\big|\,\xi_1,\dots,\xi_\ell\big] = G(h_\ell) = \mu_g^{\gamma h_\ell}(B)\,. \qquad \square$$

4.3 | Conformal quasi-invariance

Theorem 4.15. *Assume that the Riemannian manifold (M,g) is admissible and that $g' = e^{2\varphi} g$ with $\varphi \in C^\infty(M)$. For $\gamma \in (-\sqrt{2n}, \sqrt{2n})$, let $\mu_g^{\gamma h}$ and $\mu_{g'}^{\gamma h'}$ denote the plain LQG measures on (M,g) and (M,g'), resp., with $h \sim \mathsf{CGF}_{M,g}$ and $h' \sim \mathsf{CGF}_{M,g'}$. Set $v' := \mathsf{vol}_{g'}(M)$, and define a centered Gaussian random variable ξ and a function $\bar\varphi \in C^\infty(M)$ by*

$$\xi := \langle h\rangle_{g'}\,, \qquad \bar\varphi := \frac{2}{v'}\,\mathsf{k}_g(e^{n\varphi}) - \frac{1}{v'^2}\,\mathcal{K}_g(e^{n\varphi}, e^{n\varphi})\,. \tag{85}$$

Then

$$\mu_{g'}^{\gamma h'} \overset{(\mathrm{d})}{=} e^{-\gamma\xi + \frac{\gamma^2}{2}\bar\varphi + n\varphi}\,\mu_g^{\gamma h}\,. \tag{86}$$

Our formulation of the *plain* LQG measure is slightly different from the one usually considered in dimension 2, see Section 4.4 for more details.

Proof. Let $h \sim \mathsf{CGF}_{M,g}$. For $\ell \in \mathbb{N}$, let h_ℓ be the Gaussian random field defined by (66), and define the random fields

$$h'_\ell := h_\ell - \langle h_\ell\rangle_{g'}\,, \qquad h' := h - \langle h\rangle'_{g'}\,. \tag{87}$$

Random fields
The convergence $h_\ell \to h$ for $\ell \to \infty$ as stated in Proposition 3.9 implies an analogous convergence $h'_\ell \to h'$. More precisely, for every $u \in \mathring{H}_g^{n/2} = \mathring{H}_{g'}^{n/2}$,

$$\lim_{\ell\to\infty} \big\langle h_\ell\,\big|\,u\big\rangle_{g'} = \lim_{\ell\to\infty} \big\langle h_\ell\,\big|\,e^{n\varphi}u\big\rangle_g = \big\langle h\,\big|\,e^{n\varphi}u\big\rangle_g = \big\langle h\,\big|\,u\big\rangle_{g'},$$

as well as $\lim_{\ell\to\infty}\langle h_\ell\rangle_{g'} = \langle h\rangle_{g'}$, the convergences being $\mathbf{P}$-a.s. and in $L^2(\mathbf{P})$, and thus,

$$\lim_{\ell\to\infty} \big\langle h'_\ell\,\big|\,u\big\rangle_{g'} = \big\langle h'\,\big|\,u\big\rangle_{g'}, \qquad \mathbf{P}\text{-a.s. and in } L^2(\mathbf{P})\,. \tag{88}$$

Let us set $k'_{g,\ell}(x,y) := \mathbf{E}[h'_\ell(x)\,h'_\ell(y)]$ and let us denote by $\mathsf{k}'_{g,\ell}$ the corresponding integral operator on $L^2_{g'}$, namely,

$$(\mathsf{k}'_{g,\ell}u)(x) := \int k'_{g,\ell}(x,y)u(y)\mathsf{dvol}_{g'}(y)\,, \qquad u \in L^2_{g'}\,.$$

Then,

$$\begin{aligned}\lim_\ell \iint u(x)\,k'_{g,\ell}(x,y)\,v(y)\,\mathsf{dvol}_{g'}^{\otimes 2}(x,y) &= \lim_\ell \mathbf{E}\Big[\big\langle h'_\ell\,\big|\,u\big\rangle_{g'}\big\langle h'_\ell\,\big|\,v\big\rangle_{g'}\Big] \\ &= \mathbf{E}\Big[\big\langle h'\,\big|\,u\big\rangle_{g'}\big\langle h'\,\big|\,v\big\rangle_{g'}\Big] \\ &= \iint u(x)\,k_{g'}(x,y)\,v(y)\,\mathsf{dvol}_{g'}^{\otimes 2}(x,y)\,,\end{aligned}$$

where the first equality holds by definition of $k'_{g,\ell}$, the second equality holds by (88), and the third equality holds since $h' \sim \mathsf{CGF}_{M,g'}$ by Theorem 3.13. In particular, we have the following convergences:

$$\lim_\ell \sqrt{\mathsf{k}'_{g,\ell}}\, u = \sqrt{\mathsf{k}_{g'}}\, u\,, \qquad \forall u \in L^2_{g'}\,, \tag{89}$$

$$\lim_\ell k'_{g,\ell} = k_{g'}, \qquad \text{a.e. on } M \times M. \tag{90}$$

Random measures

Now, let us set, for all $\ell \in \mathbb{N}$:

$$\mu_g^{\gamma h_\ell} := e^{\gamma h_\ell - \frac{\gamma^2}{2}\mathbf{E}[h_\ell^2]}\,\mathsf{vol}_g\,, \qquad \text{resp.} \qquad \mu_{g'}^{\gamma h'_\ell} := e^{\gamma h'_\ell - \frac{\gamma^2}{2}\mathbf{E}[(h'_\ell)^2]}\,\mathsf{vol}_{g'}\,.$$

On the one hand, by Theorem 4.1, we have that

$$\lim_\ell \int u\, d\mu_g^{\gamma h_\ell} = \int u\, d\mu_g^{\gamma h}\,, \qquad u \in \mathcal{C}(M)\,, \tag{91}$$

in $L^1(\mathbf{P})$. On the other hand, similarly to the proof of Theorem 4.1, the martingale $\{\mu_{g'}^{\gamma h'_\ell}(M) : \ell \in \mathbb{N}\}$ is uniformly integrable. Together with (89) and (90), this verifies the assumptions in [78, Thm. 25], hence

$$\lim_\ell \int u\, d\mu_{g'}^{\gamma h'_\ell} = \int u\, d\mu_{g'}^{\gamma h'}\,, \qquad u \in \mathcal{C}_b(M)\,, \tag{92}$$

in $L^1(\mathbf{P})$.

Radon–Nikodym derivative

Similarly to Theorem 2.20, we can compute $k'_{g,\ell}$ explicitly. For short write $\mathsf{m}_{g'} = \mathsf{vol}_{g'}/v'$. Then, we have

$$\begin{aligned}
k'_{g,\ell}(x,y) := \mathbf{E}\big[h'_\ell(x)\, h'_\ell(y)\big] &= \mathbf{E}\big[\big(h_\ell(x) - \langle h_\ell\rangle_{g'}\big)\,\big(h_\ell(y) - \langle h_\ell\rangle_{g'}\big)\big] \\
&= k_{g,\ell}(x,y) + \iint k_{g,\ell}(w,z)\,d\mathsf{m}_{g'}^{\otimes 2}(w,z) \\
&\quad - \int k_{g,\ell}(x,z)\,d\mathsf{m}_{g'}(z) - \int k_{g,\ell}(y,w)\,d\mathsf{m}_{g'}(w) \\
&= k_{g,\ell}(x,y) - \frac{1}{2}\bar{\varphi}_\ell(x) - \frac{1}{2}\bar{\varphi}_\ell(y),
\end{aligned}$$

where we have set

$$\begin{aligned}
\bar{\varphi}_\ell(\,\cdot\,) &:= 2\int k_{g,\ell}(\,\cdot\,,z)\,d\mathsf{m}_{g'}(z) - \iint k_{g,\ell}(w,z)\,d\mathsf{m}_{g'}^{\otimes 2}(w,z) \\
&= \frac{2}{v'}\,\mathsf{k}_{g,\ell}(e^{n\varphi}) - \frac{1}{v'^2}\,\mathcal{K}_{g,\ell}(e^{n\varphi}, e^{n\varphi}).
\end{aligned}$$

Thus, in particular,

$$k'_{g,\ell}(x,x) - k_{g,\ell}(x,x) = \bar{\varphi}_\ell(x).$$

Furthermore, set $\xi_\ell := \langle h_\ell \rangle_{g'} = \langle h_\ell \,|\, e^{n\varphi} \rangle_g$. Then, almost surely:

$$\begin{aligned}\log \frac{d\mu_g^{\gamma h_\ell}}{d\mu_{g'}^{\gamma h'_\ell}}(x) &= \gamma\, h_\ell(x) - \frac{\gamma^2}{2}\mathbf{E}\left[h_\ell(x)^2\right] - \gamma\, h'_\ell(x) + \frac{\gamma^2}{2}\mathbf{E}\left[h'_\ell(x)^2\right] - n\varphi(x)\\
&= \gamma\, \langle h_\ell \rangle_{g'} + \frac{\gamma^2}{2}\mathbf{E}\left[h'_\ell(x)^2 - h_\ell(x)^2\right] - n\varphi(x)\\
&= \gamma\, \xi_\ell + \frac{\gamma^2}{2}\left(k'_\ell(x,x) - k_\ell(x,x)\right) - n\varphi(x)\\
&= \gamma\, \xi_\ell - \frac{\gamma^2}{2}\bar{\varphi}_\ell(x) - n\varphi(x),\end{aligned}$$

and thus, for every $u \in C_b(M)$,

$$\int_M u(x)\, d\mu_g^{\gamma h'_\ell}(x) = \int_M e^{-\gamma\, \xi_\ell + \frac{\gamma^2}{2}\bar{\varphi}_\ell(x) + n\varphi(x)}\, u(x)\, d\mu_{g'}^{\gamma h_\ell}(x)\ . \tag{93}$$

Convergence

As $\ell \to \infty$, by (88) applied with $u = e^{n\bar{\varphi}}$, we have that $\xi_\ell \to \xi$, $\mathbf{P}$-a.s. Moreover, $\bar{\varphi}_\ell \to \bar{\varphi}$ in $L^\infty(M, \mathrm{vol}_g)$ according to Lemma 2.15(vii). Together with the representation formula (93) and the convergence obtained in (91) and (92), this implies that, in $L^1(\mathbf{P})$,

$$\begin{aligned}\int_M u(x)\, d\mu_{g'}^{\gamma h'}(x) &= \lim_{\ell\to\infty} \int_M u(x)\, d\mu_{g'}^{\gamma h'_\ell}(x)\\
&= \lim_{\ell\to\infty} \int_M e^{-\gamma\, \xi_\ell + \frac{\gamma^2}{2}\bar{\varphi}_\ell(x) + n\varphi(x)}\, u(x)\, d\mu_g^{\gamma h_\ell}(x)\\
&= \lim_{\ell\to\infty} \int_M e^{-\gamma\, \xi + \frac{\gamma^2}{2}\bar{\varphi}(x) + n\varphi(x)}\, u(x)\, d\mu_g^{\gamma h_\ell}(x)\\
&= \int_M e^{-\gamma\, \xi + \frac{\gamma^2}{2}\bar{\varphi}(x) + n\varphi(x)}\, u(x)\, d\mu_g^{\gamma h}(x)\ .\end{aligned}$$

This proves the claim. □

Corollary 4.16. *Assume that (M, g) and (M', g') are admissible and conformally equivalent with diffeomorphism Φ and conformal weight $e^{2\varphi}$. Let h and $h,$ denote the co-polyharmonic random fields, and $\mu_g^{\gamma h}$ and $\mu_{g'}^{\gamma h'}$ the corresponding plain LQG measures on (M, g) and (M', g'), resp. Then*

$$\mu_{g'}^{\gamma h'} \overset{(\mathrm{d})}{=} \Phi_*\left(e^{-\gamma\xi + \frac{\gamma^2}{2}\bar{\varphi} + n\varphi}\, \mu_g^{\gamma h}\right) \tag{94}$$

with ξ and $\bar{\varphi}$ as above.

As for the co-polyharmonic Gaussian field, the conformal quasi-invariance simplifies whenever we consider plain LQG measures constructed from ungrounded fields. To this end, we use the identification $H_g^{-\varepsilon} \simeq \mathring{H}_g^{-\varepsilon} \oplus \mathbb{R}$ with bijections $\hbar \mapsto (\pi_g(\hbar), \langle \hbar \rangle_g)$, $(h, a) \mapsto h + a$, and extend the definition of the map $\mathring{H}_g^{-\varepsilon} \to \mathcal{M}_b(M)$, $h \mapsto \mu_g^{\gamma h}$ from (78) to a map $H_g^{-\varepsilon} \to \mathcal{M}_b(M)$ by putting

$$\mu_g^{\gamma(h+a)} := e^{\gamma a}\, \mu_g^{\gamma h} \ . \tag{95}$$

Corollary 4.17. *Assume that $\hbar \sim \widehat{\mathsf{CGF}}_g$ and $\hbar' \sim \widehat{\mathsf{CGF}}_{g'}$, then*

$$\mu_{g'}^{\gamma \hbar'} \overset{(d)}{=} e^{n\varphi + \frac{\gamma^2}{2}\bar{\varphi}} \mu_g^{\gamma \hbar}.$$

Even more, for all measurable $F : H_g^{-\varepsilon} \times \mathcal{M}_b(M) \to \mathbb{R}_+$:

$$\int F\Big(\hbar', \mu_{g'}^{\gamma \hbar'}\Big) d\widehat{\mathsf{CGF}}_{g'}(\hbar') = \int F\Big(\hbar,\, e^{n\varphi + \frac{\gamma^2}{2}\bar{\varphi}} \mu_g^{\gamma\hbar}\Big)\, d\widehat{\mathsf{CGF}}_g(\hbar) \ .$$

Proof. Expanding the definition of $\widehat{\mathsf{CGF}}$ and using Theorems 3.13 and 4.15, we find

$$\begin{aligned} \int F\Big(\hbar', \mu_{g'}^{\gamma \hbar'}\Big)\, d\widehat{\mathsf{CGF}}_{g'}(\hbar') &= \int F\Big(h' + a,\, e^{\gamma a} \mu_{g'}^{\gamma h'}\Big)\, da\, d\mathsf{CGF}_{g'}(h') \\ &= \int F\Big(h - \xi + a,\, e^{\gamma(a-\xi)} e^{\frac{\gamma^2}{2}\bar{\varphi} + n\varphi} \mu_g^{\gamma h}\Big)\, da\, d\mathsf{CGF}_g(h) \ . \end{aligned}$$

We conclude by the translation invariance of the Lebesgue measure. □

4.4 | Liouville quantum gravity measure

Recall that k_g is the kernel of the inverse of the normalized co-polyharmonic operator p_g. Now we propose a further additive normalization in terms of the function

$$r_g(x) = \limsup_{y \to x} \left[k_g(x, y) - \log \frac{1}{d_g(x, y)} \right], \qquad \forall x \in M.$$

This function has an important quasi-invariance property under conformal changes.

Lemma 4.18. *Let φ smooth and $g' = e^{2\varphi} g$. Then with the notation of Theorem 4.15,*

$$r_{g'} - r_g = -\bar{\varphi} + \varphi \ .$$

Proof. By Proposition 2.20, for $x \neq y \in M$:

$$\begin{aligned} &\big[k_{g'}(x, y) + \log d_{g'}(x, y)\big] - \big[k_g(x, y) + \log d_g(x, y)\big] \\ &\qquad = -\frac{1}{2}\bar{\varphi}(x) - \frac{1}{2}\bar{\varphi}(y) + \log d_{g'}(x, y) - \log d_g(x, y) \ . \end{aligned}$$

Thus, the claim is obtained immediately by letting $y \to x$, and noting that

$$\frac{d_{g'}(x,y)}{d_g(x,y)} \longrightarrow e^{\varphi(x)} \qquad \text{as } y \longrightarrow x \ . \qquad \square$$

Definition 4.19. We define the *LQG measure* (also called *adjusted LQG measure*) by

$$\bar{\mu}_g^{\gamma h} = \exp\left(\frac{\gamma^2}{2} r_g\right) \mu_g^{\gamma h}.$$

Theorem 4.20. *Let φ be smooth, $g' = e^{2\varphi} g$, and h and h' co-polyharmonic Gaussian fields with respect to g and g'. Then*

$$\bar{\mu}_{g'}^{\gamma h'} = \exp\left[-\gamma\xi + \left(n + \tfrac{\gamma^2}{2}\right)\varphi\right]\bar{\mu}_g^{\gamma h},$$

where $\xi = \langle h \rangle_{g'}$.

Proof. This is a direct consequence of Theorem 4.15 and Lemma 4.18. $\square$

As before, in order to get rid of the Gaussian random variable ξ in the conformal quasi-invariance formulation, we need to consider the law induced by the ungrounded co-polyharmonic field, cf. Corollary 4.17.

Corollary 4.21. *Assume that $\hbar \sim \widehat{\mathsf{CGF}}_g$ and $\hbar' \sim \widehat{\mathsf{CGF}}_{g'}$, then*

$$\bar{\mu}_{g'}^{\gamma \hbar'} \overset{(d)}{=} e^{(n+\frac{\gamma^2}{2})\varphi} \bar{\mu}_g^{\gamma \hbar} \ ,$$

or, in other words,

$$\bar{\mu}_{g'}^{\gamma \hbar'} \overset{(d)}{=} \bar{\mu}_g^{\gamma\, T(\hbar)} \ ,$$

with the shift $T : \hbar \mapsto \hbar + \left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)\varphi$.

Even more, for all measurable $F : H_g^{-\varepsilon} \times \mathcal{M}_b(M) \to \mathbb{R}_+$:

$$\int F\left(\hbar', \bar{\mu}_{g'}^{\gamma \hbar'}\right) d\widehat{\mathsf{CGF}}_{g'}(\hbar') = \int F\left(\hbar, e^{(n+\frac{\gamma^2}{2})\varphi} \bar{\mu}_g^{\gamma \hbar}\right) d\widehat{\mathsf{CGF}}_g(\hbar) \ .$$

Remark 4.22. Let us assume for the sake of discussion that the function r_g is smooth. This is known to be true in the case $n = 2$; in arbitrary even dimension, according to [66, Lem. 2.1], the function r_g is at least C^2. With respect to the smooth function r_g, we define the *refined co-polyharmonic kernel* by

$$\tilde{k}_g(x,y) := k_g(x,y) - \frac{1}{2} r_g(x) - \frac{1}{2} r_g(y) + c_g \ ,$$

where $c_g := \langle r_g \rangle_g + \frac{1}{4}\mathfrak{p}_g(r_g, r_g)$. With k_g, it shares the estimate (59), and in addition, it satisfies

$$\limsup_{y \to x} \left[\tilde{k}_g(x,y) - \log \frac{1}{d_g(x,y)}\right] = c_g \ , \qquad x \in M \ . \tag{96}$$

(i) The kernel $\tilde{k}_g$ is the covariance kernel associated with the *refined co-polyharmonic field* given by

$$\tilde{h}_g := h_g - \frac{1}{2}\left\langle h_g \,\middle|\, \mathsf{p}_g r_g \right\rangle_g ,$$

where h_g denotes the co-polyharmonic field as considered before.

(ii) Let $\tilde{k}_{g'}$ and $\tilde{h}_{g'}$ denote the refined kernel and refined field associated with the metric $g' = e^{2\varphi} g$ for some $\varphi \in C^\infty(M)$. Then

$$\tilde{k}_{g'}(x,y) = \tilde{k}_g(x,y) + \frac{1}{2}\varphi(x) - \frac{1}{2}\varphi(y) + c_{g'} - c_g , \tag{97}$$

and

$$\tilde{h}_{g'} \overset{(d)}{=} \tilde{h}_g - \frac{1}{2}\langle \tilde{h}_g \,|\, \mathsf{p}_g \varphi\rangle_g . \tag{98}$$

(iii) The LQG measure (aka Gaussian multiplicative chaos) on (M, g) associated with the refined field $\tilde{h}_g$ is given in terms of the plain LQG measure associated with h_g as

$$\mu_g^{\gamma\tilde{h}} = \exp\left(\frac{\gamma^2}{2}(r_g - c_g) - \frac{\gamma}{2}\left\langle h \,\middle|\, \mathsf{p}_g r_g \right\rangle_g\right) \mu_g^{\gamma h} .$$

Passing from grounded polyharmonic fields $h \sim \mathsf{CGF}_g$ to ungrounded fields $\hbar = h + a \sim \mathsf{CGF}_g \otimes \mathfrak{L}^1$ and making use of the translation invariance of $\mathfrak{L}^1$ on $\mathbb{R}$, the associated LGQ measures satisfy

$$\mu_g^{\gamma\tilde{\hbar}} = e^{\gamma a}\, \mu_g^{\gamma\tilde{h}} \overset{(d)}{=} \exp\left(\frac{\gamma^2}{2} r_g\right) e^{\gamma a}\, \mu_g^{\gamma h} = e^{\gamma a}\, \bar{\mu}_g^{\gamma h} = \bar{\mu}_g^{\gamma \hbar}. \tag{99}$$

That is, the plain LGQ measure for the ungrounded refined co-polyharmonic field coincides in distribution with the adjusted LGQ measure for the ungrounded co-polyharmonic field.

Working with the adjusted LGQ measure for the co-polyharmonic field (rather than with the plain LGQ measure for the refined co-polyharmonic field) allows us to avoid any smoothness assumption on r_g.

Proof.

(i) Straightforward calculations yield for $u, v \in \mathring{H}_g^{n/2}$,

$$\begin{aligned}
&\iint \tilde{k}_g(x,y)\, u(x)\, v(y)\, \mathsf{dvol}_g(x)\, \mathsf{dvol}_g(y) \\
&\quad = \mathbf{E}\left[\langle \tilde{h} \,|\, u\rangle_g \cdot \langle \tilde{h} \,|\, v\rangle_g\right] \\
&\quad = \iint k_g(x,y) u(x) v(y) \mathsf{dvol}_g(x)\, \mathsf{dvol}_g(y) \\
&\qquad - \frac{1}{2}\int u\, \mathsf{dvol}_g \cdot \mathcal{K}_g(\mathsf{p}_g r_g, v) - \frac{1}{2}\int v\, \mathsf{dvol}_g \cdot \mathcal{K}_g(\mathsf{p}_g r_g, u) \\
&\qquad + \frac{1}{4}\int u\, \mathsf{dvol}_g \cdot \int v\, \mathsf{dvol}_g \cdot \mathcal{K}_g(\mathsf{p}_g r_g, \mathsf{p}_g r_g) .
\end{aligned}$$

(ii) Immediate consequences of Lemma 4.18, Proposition 2.20, and Theorem 4.15. □

Remark 4.23. Most approaches to the LQG measure in dimension 2 are formulated in terms of a ("background") metric tensor g that is translation invariant. In these cases, $r_g \equiv const$ and thus plain and adjusted LQG measure coincide up to a multiplicative constant. Both measures are obtained by regularizing h via convolution and by normalizing $e^{\gamma h_\varepsilon(x)}\mathrm{vol}(dx)$ by some explicit power of ε, say $\varepsilon^{\gamma^2/2}$. This translation invariant renormalization in terms of $\varepsilon^{\gamma^2/2}$ is then also employed in the nonhomogeneous case ([28, sect. 2.3]: "We need that these cut-off approximations be defined with respect to a fixed background metric: we consider Euclidean circle averages of the field because they facilitate some computations…"), whereas the "intrinsic Riemannian perspective" would suggest to renormalize by $\exp\left(-\gamma^2/2\,\mathbf{E}[h_\varepsilon(x)^2]\right)$. As observed in [28, Prop. 2.5] and [47, Lemma 3.2], in our notation

$$\mathbf{E}[h_\varepsilon(x)^2] = -\log(\varepsilon) + r_g(x) + o(1),$$

and thus, the LQG measure constructed in [28] and [47] for arbitrary closed Riemannian surfaces coincides with the adjusted LQG measure in our sense.

5 | LIOUVILLE BROWNIAN MOTION AND RANDOM GJMS OPERATORS

5.1 | Support properties of LQG measures

In the sequel, we will study support properties on M for a.e. realization of LQG measures. Since the (adjusted) LQG measure $\bar{\mu}_g^{\gamma h}$ and the plain LQG measure $\mu_g^{\gamma h}$ only differ by a deterministic (finite and positive) weight, these support properties will be the same for both of them. For convenience, we will state them for the plain LQG measure.

Since a typical realization of the plain LQG measure $\mu_g^{\gamma h}$ is singular with respect to the volume measure of M, it gives positive mass to certain sets $E \subset M$ of vanishing volume measure. However, it does not give mass to sets of vanishing $\mathcal{H}^s$-capacity (for sufficiently large s), a classical scale of "smallness of sets" involving Green kernels and thus well suited for our purpose.

Definition 5.1. For $s > 0$, the *$\mathcal{H}^s$-capacity* (aka *Bessel capacity*) of a set $E \subset M$ is

$$\mathrm{cap}_s(E) := \inf\left\{ \|f\|_{L^2}^2 \ :\ \mathsf{G}_{s/2,1} f \geqslant 1 \ \mathrm{vol}_g\text{-a.e. on } E,\ f \geqslant 0 \right\}. \tag{100}$$

A set with vanishing $\mathcal{H}^s$-capacity, also has vanishing $\mathcal{H}^r$-capacity for every $r \in (0, s)$, we call a set E such that $\mathrm{cap}_s(E) = 0$, a cap_s*-zero* or a cap_s*-polar* set.

Theorem 5.2. *Consider the co-polyharmonic Gaussian field $h \sim \mathsf{CGF}_g$ and the associated plain LQG measure $\mu_g^{\gamma h}$ on (M, g) with $|\gamma|^2 < 2n$. Then for a.e. h and every $s > \gamma^2/4$, the measure $\mu_g^{\gamma h}$ does not charge sets of vanishing $\mathcal{H}^s$-capacity. That is,*

$$\mathit{cap}_s(E) = 0 \implies \mu_g^{\gamma h}(E) = 0 \qquad \textit{for every Borel } E \subset M\ .$$

For applications of this result in the remainder of this paper, two choices of s are relevant, $s = n/2$ and $s = 1$.

Corollary 5.3. *Consider h and $\mu_g^{\gamma h}$ as above.*

- *If $|\gamma| < \sqrt{2n}$, then $\mathbf{P}$-almost surely $\mu_g^{\gamma h}$ does not charge sets of vanishing $\mathcal{H}^{n/2}$-capacity.*
- *If $|\gamma| < 2$, then $\mathbf{P}$-almost surely $\mu_g^{\gamma h}$ does not charge sets of vanishing $\mathcal{H}^1$-capacity.*

In the particular case $n = 2$, both assertions coincide. In general, none of the two assertions is an immediate consequence of the other one.

Our proof of the theorem relies on results on Bessel capacities and on a celebrated estimate for the volume of balls by J.-P. Kahane for random measures defined in terms of covariance kernels with logarithmic divergence.

Concerning capacities, we adapt to manifolds results in [92] that do not follow from [27]. Denote by $\mathcal{M}_b(M)$ the space of nonnegative finite Borel measures on M. For $\mu \in \mathcal{M}_b(M)$, we set

$$\mathsf{G}_{s,\alpha}\mu(x) = \int G_{s,\alpha}(x,y)d\mu(y), \qquad s,\alpha > 0 \ ,$$

and, for a measurable set $E \subset M$:

$$b_s(E) := \sup\left\{\mu(E) \, : \, \mu \in \mathcal{M}_b(M), \ \left\|\mathsf{G}_{s/2,1}(\mathbf{1}_E\mu)\right\|_{L^2} \leqslant 1\right\}. \tag{101}$$

Remark 5.4. The Bessel capacities as defined above are “order 1 capacities” in the sense of Dirichlet forms. The corresponding “order 0 capacities” would be defined by replacing the operator $\mathsf{G}_{s,1}$ with its grounded version $\mathring{\mathsf{G}}_s$. As a consequence of the compactness of M, these capacities define the same class of cap-zero subsets of M.

Lemma 5.5. *Let $s > 0$. The following assertions hold true:*

(i) cap_s *is a regular Choquet capacity;*
(ii) *for every Suslin set $E \subset M$,*

$$b_s(E)^2 = cap_s(E) \, ;$$

(iii) *if $\mu \in \mathcal{M}_b(M)$ satisfies $\|\mathsf{G}_{s/2,1}\mu\|_{L^2} < \infty$, then μ does not charge cap_s-zero sets;*
(iv) *any function in $\mathcal{H}^s$ is pointwise determined (and finite) up to a cap_s-zero set;*
(v) *if $(u_k)_k \subset \mathcal{H}^s$ and $u \in \mathcal{H}^s_g$ satisfy $\lim_k \left|u_k - u\right|_{\mathcal{H}^s} = 0$, then there exists a subsequence $(u_{k_j})_j \subset \mathcal{H}^s$ so that $u = \lim_j u_{k_j}$ pointwise up to a cap_s-zero set.*

Proof. Since assertions (i) and (ii) above are set-theoretical in nature, their proof is adapted *verbatim* from [92]. In particular, (i) is concluded as in [92, Cor. 2.6.9], and (ii) as in [92, Thm. 2.6.12]. These adaptations hold provided we substitute the operator $g_\alpha\, *$ in [92], $\alpha = s/2$, with $\mathsf{G}_{s/2,1}$, and noting that $G_{s/2,1}(x, \cdot)$ is continuous away from x.

In order to show (iii), let $E \subset M$ be cap_s-polar. By standard facts on Choquet capacities, E can be covered by countably many Suslin cap_s-polar sets. Thus, we may assume with no loss of generality that E be additionally Suslin. By (ii) and definition (101) of b_s, we then have

$$\left\|\mathsf{G}_{s/2,1}\mu\right\|_{L^2} \cdot \mathrm{cap}_s(E)^{1/2} \geqslant \left\|\mathsf{G}_{s/2,1}(\mathbf{1}_E\mu)\right\|_{L^2} \cdot b_s(E) \geqslant \mu E \ ,$$

which concludes the proof by assumption on E.

(iv) By definition, u is in $\mathcal{H}^s$ if and only if there exists $v \in L^2$ such that $u = \mathsf{G}_{s/2,1}v$, and $\|u\|_{\mathcal{H}^s} = \|v\|_{L^2}$. By density of $C^\infty(M)$ in both L^2 and $\mathcal{H}^s$, it suffices to show that, whenever $u_n \to u$ vol_g-a.e. and in L^2, then $\mathsf{G}_{s/2,1}u_n \to \mathsf{G}_{s/2,1}u$ cap_s-q.e. This latter fact holds as in [92, Lem. 2.6.4], with identical proof.

(v) First, let us show that

$$\mathrm{cap}_s(\{|f| > a\}) \leqslant \|f\|^2_{\mathcal{H}^s}/a^2 \ , \qquad f \in \mathcal{H}^s \ , \ a > 0 \ . \tag{102}$$

Indeed,

$$\mathsf{G}_{s/2,1}(1-\Delta_g)^{s/2}|f|/a = |f|/a \geqslant 1 \quad \text{on } \{|f| > a\} \ ,$$

hence, by definition of cap_s, we have that

$$\mathrm{cap}_s(\{|f| > a\}) \leqslant \left\|(1-\Delta_g)^{s/2}|f|/a\right\|^2_{L^2} = \left\||f|/a\right\|^2_{\mathcal{H}^s} = \|f\|^2_{\mathcal{H}^s}/a^2 \ .$$

Now, let $(u_{k_j})_j \subset (u_k)_k$ be so that $\|u - u_{k_j}\|^2_{\mathcal{H}^s} \leqslant 2^{-3j}$, and set $A_j := \{|u - u_{k_j}| > 2^{-j}\}$. By (102),

$$\mathrm{cap}_s(A_j) \leqslant 2^{2j}\|u - u_{k_j}\|^2_{\mathcal{H}^s} \leqslant 2^{-j} \ .$$

Set $A := cap_{\ell=1}^\infty \bigcup_{j=\ell}^\infty A_j$. If $x \notin A$, it is readily seen that $\lim_j |u(x) - u_{k_j}(x)| = 0$ by definition of the sets A_j. Thus, it suffices to show that $\mathrm{cap}_s(A) = 0$. Since cap_s is a Choquet capacity, it is increasing and (countably) subadditive, and we have that

$$\mathrm{cap}_s(A) \leqslant \mathrm{cap}_s\left(\bigcup_{j=\ell}^\infty A_j\right) \leqslant \sum_{j=\ell}^\infty \mathrm{cap}_s(A_j) \leqslant \sum_{j=\ell}^\infty 2^{-j} = 2^{-\ell+1} \ , \qquad \ell \in \mathbb{N} \ .$$

Since ℓ was arbitrary, the conclusion follows letting $\ell \to \infty$. □

For the above-mentioned, celebrated estimate for the volume of balls by J.-P. Kahane — concerning the so-called R^+_α classes —, we refer to the survey [75] by R. Rhodes and V. Vargas. Note that $|k_g(x,y) + \log d(x,y)| \leqslant C$ and recall Remark 3.8 concerning positivity of k_g.

Lemma 5.6 [75, Thm. 2.6]. *Take $\alpha \in (0,n)$ and $\gamma^2/2 \leqslant \alpha$. Consider a co-polyharmonic Gaussian field $h \sim \mathsf{CGF}_g$ and the associated plain LQG measure $\mu_g^{\gamma h}$ on (M,g). Then almost surely, for all $\varepsilon > 0$, there exists $\delta > 0$, $C < \infty$, and a compact set $M_\varepsilon \subset M$ such that $\mu_g^{\gamma h}(M \setminus M_\varepsilon) < \varepsilon$ and*

$$\mu_g^{\gamma h}(B_r(x) \cap M_\varepsilon) \leqslant C r^{\alpha - \gamma^2/2 + \delta}, \qquad \forall r > 0, \forall x \in M. \tag{103}$$

Proof of Theorem 5.2. For a.e. h, the following holds true. Let numbers $\gamma, s \in \mathbb{R}$ with $\gamma^2 < 4s \leqslant 2n$ be given as well as a Borel set $E \subset M$ with $\mu_g^{\gamma h}(E) > 0$. Applying Lemma 5.6 with $\alpha := n + \gamma^2/2 - 2s \in [\gamma^2/2, n)$ and $\varepsilon := \frac{1}{2}\mu_g^h(E) > 0$ yields the existence of $\delta > 0$, $C < \infty$, and a compact set $M_\varepsilon \subset M$ such that $\mu_g^{\gamma h}(M \setminus M_\varepsilon) < \varepsilon$ and (103) holds. Set $\mu_\varepsilon := \mathbf{1}_{M_\varepsilon}\mu_g^{\gamma h}$. Then, $\mu_\varepsilon(E) \geqslant \varepsilon > 0$. Furthermore, with $f(r) := r^{2s-n}$ and $R := \mathrm{diam}(M)$, uniformly in y,

$$\begin{aligned}\int_M f(d(x,y))\, d\mu_\varepsilon(x) &= -\int_M \int_0^R \mathbf{1}_{\{r > d(x,y)\}} f'(r) dr\, d\mu_\varepsilon(x) \\ &= -\int_0^R \mu_\varepsilon(B_r(y))\, f'(r)\, dr\end{aligned}$$

$$\leqslant (n-2s)\int_0^R r^{\alpha-\gamma^2/2+\delta}\, r^{2s-n}\frac{dr}{r}$$
$$= (n-2s)\int_0^R r^{\delta}\frac{dr}{r} \leqslant C' < \infty\ .$$

Hence, according to Lemma 2.10,

$$0 \leqslant G_{s,1}\mu_\varepsilon(y) \leqslant C'\ . \tag{104}$$

Thanks to the convolution property of the kernels $G_{r,1}$ for $r>0$ [29, Lem. 2.3(*ii*)], the uniform estimate (104), and the fact that $\mu_g^{\gamma h}$ is a finite measure, we find, with $\mu' := \mathbf{1}_E\mu_\varepsilon$:

$$\begin{aligned}\left\|G_{s/2,1}(\mu')\right\|_{L^2}^2 &= \iiint G_{s/2,1}(x,y)\,d\mu'(y)\,G_{s/2,1}(x,z)\,d\mu'(z)\,\mathrm{dvol}_g(x)\\ &= \iint G_{s,1}(y,z)\,d\mu'(y)\,d\mu'(z)\\ &\leqslant C'\cdot\mu'(M) =: C'' < \infty.\end{aligned}$$

Hence, by the very definition of b_s,

$$b_s(E) \geqslant \frac{\mu'(E)}{\|G_{s/2,1}(\mu')\|_{L^2}} \geqslant \frac{\varepsilon}{\sqrt{C''}} > 0\,,$$

and thus, in turn, $\mathrm{cap}_s(E) > 0$ according to Lemma 5.5. □

5.2 | Random Dirichlet form and Liouville Brownian motion

For sufficiently small $|\gamma|$, the LQG measure $\mu_g^{\gamma h}$ does not charge sets of $\mathcal{H}^1$-capacity zero. Hence, a random Brownian motion can easily be constructed through time change of the standard Brownian motion $((B_t)_{t\geqslant 0}, (\mathbb{P}_x)_{x\in M})$.

Theorem 5.7. *Let (M,g) be admissible, let $h \sim \mathsf{CGF}_g$ denote the co-polyharmonic Gaussian field, and $\mu_g^{\gamma h}$ the associated LQG measure with $|\gamma| < 2$. Then, for* $\mathbf{P}$*-a.e. h,*

(i) *A regular strongly local Dirichlet form on $L^2(M,\mu_g^{\gamma h})$ is given by*

$$\mathcal{E}^h(f,f) := \int_M |\nabla f|^2\,\mathrm{dvol}_g\,,\qquad \mathcal{D}(\mathcal{E}^h) := \left\{ f\in\mathcal{H}^1(M)\,:\ \tilde f\in L^2(M,\mu_g^{\gamma h})\right\} \tag{105}$$

where $\tilde f$ denotes the quasi-continuous modification of $f\in\mathcal{H}^1(M)$.

(ii) *The associated reversible continuous Markov process $((X_t^h)_{t\geqslant 0}, (\mathbb{P}_x^h)_{x\in M})$, called* Liouville Brownian motion on *(M,g), is obtained by time change of the standard Brownian motion on (M,g). Namely, let $(A_t^h)_{t\geqslant 0}$ be the additive functional whose Revuz measure is given by $\mu_g^{\gamma h}$, then*

$$\mathbb{P}_x^h := \mathbb{P}_x,\qquad X_t^h := B_{\tau_t^h}\ ,\qquad \tau_t^h := \inf\{s\geqslant 0\,:\,A_s^h > t\}\ .$$

(iii) *Moreover, for every bounded probability density ρ on M, the additive functional $(A_t^h)_{t\geqslant 0}$ is $\mathbb{P}_\rho$-a.s. given by*

$$A_t^h = \lim_{\ell\to\infty} \int_0^t \exp\left(\gamma\, h_\ell(B_s) - \frac{\gamma^2}{2} k_\ell(B_s, B_s)\right) ds\ , \tag{106}$$

with h_ℓ and k_ℓ as in (66) and (67).

Remark 5.8. Recall that the additive functional A^h associated with the measure $\mu_g^{\gamma h}$ is the process characterized by

$$\mathbb{E}_x\left[\int_0^t u(B_s) dA_s^h\right] = \int_0^t \int u(y)\, p_s(x,y)\, d\mu_g^{\gamma h}(y) ds\ , \qquad u \in \mathfrak{B}_b\ ,\ t \geqslant 0\ , \tag{107}$$

where $\mathfrak{B}_b$ denotes the space of real-valued bounded Borel functions on M. For further information on additive functionals, see [38].

Proof. (i) and (ii) hold using standard argument in the theory of Dirichlet forms. Indeed, Corollary 5.3 and the compactness of M imply that for **P**-a.e. h, the measure $\mu_g^{\gamma h}$ is a Revuz measure of finite energy integral. For details, see [45, Thm. 1.7], where this argument is carried out in the case when M is the unit disk.

(iii) Fix $t > 0$ and ρ a probability measure with bounded density on M. We consider the occupation measure

$$dL_t(x) = \int_0^t d\delta_{B_s}(x) ds\ .$$

Observe that for $\alpha > 0$,

$$\begin{aligned}
\mathbb{E}_\rho\left[\iint \frac{dL_t(y)\, dL_t(z)}{d(y,z)^\alpha}\right] &= \int_0^t\int_0^t \mathbb{E}_\rho d(B_r, B_s)^{-\alpha} dr\, ds \\
&= 2\int_0^t\int_s^t \iiint d(y,z)^{-\alpha} p_s(x,y)\, p_{r-s}(y,z) d\mathrm{vol}_g(y) d\mathrm{vol}_g(z) d\rho(x) dr ds \\
&\leqslant C\, t \sup_\ell \int_0^t \iint d(y,z)^{-\alpha} p_s(y,z)\, d\mathrm{vol}_g(y)\, d\mathrm{vol}_g(z)\, ds \\
&\leqslant C\, t e^t \iint d(y,z)^{-\alpha} G_{1,1}(y,z)\, d\mathrm{vol}_g(y)\, d\mathrm{vol}_g(z)\ .
\end{aligned}$$

According to the estimate for the 1-Green kernel $G_{1,1}$, the latter integral is finite for all $\alpha < 2$. This means that, $\mathbb{P}_\rho$-almost surely, L_t satisfies [54, Eqn. (39)] for all $\alpha < 2$. Thus, L_t is, $\mathbb{P}_\rho$-almost surely, in the class $M_{\alpha+}^+$ for all $\alpha < 2$. Arguing as in the proof of Theorem 4.1, we find that, for all $\gamma^2 < 4$, having fixed the randomness with respect to $\mathbb{P}_\rho$, there exists a random measure $\nu_t^{\gamma h}$ that is the Gaussian multiplicative chaos over $(h, \gamma k)$ with respect to L_t.

Now, for all Borel sets $A \subset M$, we set

$$\nu_t^{\gamma h_\ell}(A) := \int_A \exp\left(\gamma h_\ell(x) - \tfrac{\gamma^2}{2} k_\ell(x,x)\right) dL_t(x)$$
$$= \int_0^t 1_A(B_s) \exp\left(\gamma h_\ell(B_s) - \tfrac{\gamma^2}{2} k_\ell(B_s, B_s)\right) ds \,.$$

Since we choose $(h_\ell)_\ell$ as in (66), the family $(\nu_t^{\gamma h_\ell})_\ell$ is a $\mathbf{P}$-martingale for every fixed $t \geqslant 0$, similarly to Theorem 4.14. The fact that $\nu_t^{\gamma h_\ell} \to \nu_t^{\gamma h}$ follows from the same uniform integrability argument for martingales as in Theorem 4.14.

For all $t > 0$, set $A_t^h := \nu_t^{\gamma h}(M)$ and $A_t^{h_\ell} := \nu_t^{\gamma h_\ell}(M)$ for each $\ell \in \mathbb{N}$. It is clear that $t \mapsto A_t^{h_\ell}$ is the positive continuous additive functional associated to $\mu_g^{\gamma h_\ell}$ by the Revuz correspondence, that is, (cf. (107)),

$$\mathbb{E}_\rho\left[\int_0^t u(B_s) dA_s^{h_\ell}\right] = \int_0^t \int \left[\int p_s(x,y) u(y) d\mu_g^{\gamma h_\ell}(y)\right] d\rho(x) ds \,, \qquad u \in \mathfrak{B}_b \;,\; t \geqslant 0 \,. \tag{108}$$

Now let $\tilde{A}_t^h$ denote the positive continuous additive functional associated with $\mu_g^{\gamma h}$. Then applying (82) twice — to $\mu_g^{\gamma h_\ell} \to \mu_g^{\gamma h}$ and to $\nu_t^{h_\ell} \to \nu_t^h$ — we obtain that in $\mathbf{P}$-probability:

$$\begin{aligned}\mathbb{E}_\rho\left[\int_0^t u(B_s) dA_s^h\right] &= \mathbb{E}_\rho\left[\int_M u \, d\nu_t^h\right] \\ &= \lim_{\ell\to\infty} \mathbb{E}_\rho\left[\int_M u \, d\nu_t^{h_\ell}\right] = \lim_{\ell\to\infty} \mathbb{E}_\rho\left[\int_0^t u(B_s) dA_s^{h_\ell}\right] \\ &= \lim_{\ell\to\infty} \int_0^t \int \left[\int p_s(x,y) u(y) d\mu_g^{\gamma h_\ell}(y)\right] d\rho(x) ds \\ &= \int_0^t \int \left[\int p_s(x,y) u(y) d\mu_g^{\gamma h}(y)\right] d\rho(x) ds = \mathbb{E}_\rho\left[\int_0^t u(B_s) d\tilde{A}_s^h\right] .\end{aligned}$$

This shows that $A^h = \tilde{A}^h$ a.s. w.r.t. $\mathbf{P} \otimes \mathbb{P}_\rho$ and concludes the proof. □

Remark 5.9. The intrinsic distance associated to the Dirichlet form (105) vanishes identically. This can be easily verified, exactly as in [45, Prop. 3.1].

Remark 5.10. The previous constructions work equally well with the adjusted Liouville measure $\bar{\mu}_g^{\gamma h}$ (or with the refined Liouville measure $\tilde{\mu}_g^{\gamma h}$) in the place of the plain Liouville measure μ_g^h. For a.e. h, the resulting process, the adjusted (or refined, resp.) Brownian motion, can be regarded as the plain Brownian motion with drift.

Remark 5.11. In the case $n = 2$, Liouville Brownian motion shares an important quasi-invariance property under conformal transformations. In higher dimensions, no such — or similar — conformal quasi-invariance property holds true. Indeed, the generator of the Brownian motion, the Laplace–Beltrami operator, is quasi-invariant under conformal transformations if and only if $n = 2$.

For the two-dimensional counterparts of the previous theorem, see [6] and [45, 46].

5.3 | Random Paneitz and random GJMS operators

In higher dimensions, from the perspective of conformal quasi-invariance, the natural random operators to study are random perturbations of the co-polyharmonic operators P_g. To simplify notation, we henceforth write P and vol rather than P_g and vol_g.

Theorem 5.12. *Let (M, g) be admissible, let $h \sim \mathsf{CGF}_g$ denote the co-polyharmonic Gaussian field, and $\mu_g^{\gamma h}$ the associated plain LQG measure with $|\gamma| < \sqrt{2n}$. Then, for $\mathbf{P}$-a.e. h,*

$$\mathfrak{E}^h(u, v) := \int_M \sqrt{\mathsf{P}}u \sqrt{\mathsf{P}}v \, d\mathsf{vol}_g \,, \qquad u, v \in \mathcal{D}(\mathfrak{E}^h) := \mathcal{H}^{n/2} \cap L^2(M, \mu_g^{\gamma h}) \,,$$

is a well-defined nonnegative closed symmetric bilinear form on $L^2(M, \mu_g^{\gamma h})$.

Proof. Since $\mu_g^{\gamma h}$ does not charge $\mathrm{cap}_{n/2}$-polar sets by Theorem 5.2, and since every $f \in \mathcal{H}^{n/2}$ is $\mathrm{cap}_{n/2}$-q.e. finite by Proposition 5.5(iv), every $f \in \mathcal{H}^{n/2}$ admits a $\mu_g^{\gamma h}$-a.e. finite representative (possibly depending on h). Thus, $\mathfrak{E}^h$ is well defined on $\mathcal{H}^{n/2} \cap L^0(\mu_g^{\gamma h})$. In order to show that $\mathfrak{E}^h$ is finite on $\mathcal{D}(\mathfrak{E}^h)$, let $u = \mathsf{G}_{n/4,1} u'$, resp. $v = \mathsf{G}_{n/4,1} v' \in \mathcal{H}^{n/2}$, with $u', v' \in L^2$, and note that

$$\begin{aligned} \mathfrak{E}^h(u, v) = \left\langle \mathsf{G}_{n/4,1} u' \,\middle|\, \mathsf{P}\mathsf{G}_{n/4,1} v' \right\rangle_{L^2} &= \left\langle u' \,\middle|\, \mathsf{G}_{n/4,1}\mathsf{P}\mathsf{G}_{n/4,1} v' \right\rangle_{L^2} \\ &\leqslant \|u'\|_{L^2} \|v'\|_{L^2} \left\| \mathsf{G}_{n/4,1}\mathsf{P}\mathsf{G}_{n/4,1} \right\|_{L^2 \to L^2} < \infty \end{aligned}$$

by admissibility of M.

In order to show closedness, it suffices to show that $\mathcal{D}(\mathfrak{E}^h)$ is complete in the graph-norm

$$\|u\|_{\mathcal{D}(\mathfrak{E}^h)} := \left(\mathfrak{E}^h(u) + \|u\|^2_{L^2(\mu_g^{\gamma h})} \right)^{1/2} , \qquad u \in \mathcal{D}(\mathfrak{E}^h) \,.$$

Since $\mathfrak{E}^h$ vanishes on constant functions by Theorem 1.3(ii), it suffices to show that $\mathcal{D}(\mathring{\mathfrak{E}}^h) := \mathring{\mathcal{H}}^{n/2} \cap L^2(\mu_g^{\gamma h})$ is complete in the same norm. To this end, let $(u_k)_k$ be $\mathcal{D}(\mathring{\mathfrak{E}}^h)$-Cauchy and note that it is in particular both $L^2(\mu_g^{\gamma h})$- and $\mathfrak{E}^h$-Cauchy. In particular, there exists the $L^2(\mu_g^{\gamma h})$-limit u of $(u_k)_k$, and, up to passing to a suitable nonrelabeled subsequence, we may further assume with no loss of generality that $\lim_k u_k = u$ $\mu_g^{\gamma h}$-a.e. Furthermore, by Lemma 2.15(ii), $\mathfrak{E}^h$ defines a norm on $\mathring{\mathcal{H}}^{n/2}$, bi-Lipschitz equivalent to the standard norm of $\mathring{\mathcal{H}}^{n/2}$. As consequence, $(u_k)_k$ is as well $\mathring{\mathcal{H}}^{n/2}$-Cauchy, and, by completeness of the latter, it admits an $\mathring{\mathcal{H}}^{n/2}$-limit u'. Up to passing to a suitable nonrelabeled subsequence, by Proposition 5.5(v), we may further assume with no loss of generality that $\lim_k u_k = u'$ $\mathrm{cap}_{n/2}$-q.e. In particular, again since $\mu_g^{\gamma h}$ does not charge $\mathrm{cap}_{n/2}$-polar sets, we have that $\lim_k u_k = u$ $\mu_g^{\gamma h}$-a.e., that is $u' = u$ $\mu_g^{\gamma h}$-a.e., hence as elements of $L^2(\mu_g^{\gamma h})$. It follows that $L^2(\mu_g^{\gamma h})$-$\lim_k u_k = u$, which concludes the proof of completeness.

Nonnegativity is a consequence of the admissibility of M. Symmetry follows from that of P, Theorem 1.3(iv). □

Corollary 5.13. *Let (M, g), h, γ, and $\mu_g^{\gamma h}$ be as above. Then for $\mathbf{P}$-a.e. h, there exists a unique nonnegative self-adjoint operator P^h on $L^2(M, \mu_g^{\gamma h})$, called* random co-polyharmonic operator *or* random

GJMS operators, *defined by* $\mathcal{D}(\mathsf{P}^h) \subset \mathcal{D}(\mathfrak{E}^h)$ *and*

$$\mathfrak{E}^h(u,v) = \int u\,\mathsf{P}^h v\, d\mu_g^{\gamma h}\ , \qquad u \in \mathcal{D}(\mathsf{P}^h)\ ,\ v \in \mathcal{D}(\mathfrak{E}^h)\ .$$

In the case $n = 4$, the operators P^h are also called *random Paneitz operators*.

Corollary 5.14. *With* (M,g)*,* h*,* γ*, and* $\mu_g^{\gamma h}$ *as above, for a.e.* h*, there exists a semigroup* $(e^{-t\mathsf{P}^h})_{t>0}$ *of bounded symmetric operators on* $L^2(M,\mu_g^{\gamma h})$*, called* random co-polyharmonic heat semigroup.

Proposition 5.15. *The random co-polyharmonic heat flow* $(t,u) \mapsto e^{-t\mathsf{P}^h}u$ *is the EDE-gradient flow for* $\frac{1}{2}\mathfrak{E}^h$ *on* $L^2(M,\mu_g^{\gamma h})$*.*

Here "EDE" stands for gradient flow in the sense of "energy-dissipation-equality," see [1, Dfn. 3.4].

Proof. The energy decays along the flow according to

$$\frac{d}{dt}\mathfrak{E}^h(u_t) = \left\langle \sqrt{\mathsf{P}^h}\frac{d}{dt}u_t \,\middle|\, \sqrt{\mathsf{P}^h}u_t \right\rangle_{L^2(\mu_g^{\gamma h})} = -\left\langle \mathsf{P}^h u_t \,\middle|\, \mathsf{P}^h u_t \right\rangle_{L^2(\mu_g^{\gamma h})} = -\|\mathsf{P}^h u_t\|^2_{L^2(\mu_g^{\gamma h})}$$

for $u_t := e^{-t\mathsf{P}^h}u_0$. Moreover, for each differentiable curve v_t, we have

$$\frac{d}{dt}\mathfrak{E}^h(v_t) = \frac{d}{dt}\int \sqrt{\mathsf{P}^h}v_t\sqrt{\mathsf{P}^h}v_t\, d\mu_g^{\gamma h} = \int \frac{d}{dt}v_t \mathsf{P}^h v_t\, d\mu_g^{\gamma h} = \left\langle \frac{d}{dt}v_t \,\middle|\, \mathsf{P}^h v_t \right\rangle_{L^2}. \tag{109}$$

Consequently $\nabla\mathfrak{E}^h(v) = \mathsf{P}^h v$ which leads to

$$\frac{d}{dt}u_t = -\nabla\mathfrak{E}^h(u_t) \tag{110}$$

and thus the assertion. □

Remark 5.16. For a.e. h and every $u \in L^2(M,\mu_g^{\gamma h})$, the solutions $u_t := e^{-t\mathsf{P}^h}u$ are absolutely continuous with respect to $\mu_g^{\gamma h}$ for all $t > 0$. It is plausible to conjecture that there exists a *random co-polyharmonic heat kernel* p_t^h such that

$$e^{-t\mathsf{P}^h}u(x) = \int_M p_t^h(x,y)\,u(y)\,d\mu_g^{\gamma h}(y) \qquad \text{for a.e. } x \in M\ , \qquad u \in L^2\ .$$

In the case $n = 2$, such a kernel exists, and it admits sub-Gaussian upper bounds (see [2, 62]),

$$p_t^h(x,y) \leqslant C_1 t^{-1}\log(t^{-1})\exp\left(-C_2\left(\frac{d(x,y)^\beta \wedge 1}{t}\right)^{\frac{1}{\beta-1}}\right), \qquad t \in \left(\tfrac{1}{2},1\right]\ ,$$

for any $\beta > \frac{1}{2}(\gamma+2)^2$ and constants $C_i = C_i(\beta,\gamma,h,d(y,0))$.

Now let us address the conformal quasi-invariance of the random co-polyharmonic operators. For this purpose, of course, we have to emphasize all g-dependencies in the notation and thus write P_g and P_g^h rather than P and P^h.

Assume that the Riemannian manifold (M, g) is admissible and that $|\gamma| < \sqrt{2n}$. Let $h \sim \mathsf{CGF}_g$ denote the co-polyharmonic random field and $\mu_g^{\gamma h}$ the corresponding plain LQG measure on (M, g).

Given any $g' = e^{2\varphi} g$ with $\varphi \in \mathcal{C}^\infty(M)$, define (a version of) the LQG measure on (M, g') according to Theorem 4.15 by

$$\mu_{g'}^{\gamma h} := e^F \mu_g^{\gamma h} . \tag{111}$$

with $v' = \mathsf{vol}_{g'}(M)$ and

$$F := -\gamma \langle h \rangle_{g'} + \frac{\gamma^2}{2v'} \mathcal{K}_g(e^{n\varphi}, e^{n\varphi}) - \left(\frac{\gamma}{v'}\right)^2 \mathsf{k}_g(e^{n\varphi}) + n\varphi . \tag{112}$$

Theorem 5.17. *The random co-polyharmonic operator P_g^h is conformally quasi-invariant: if $g' = e^{2\varphi} g$, then*

$$\mathsf{P}_{g'}^{h'} \overset{(\mathrm{d})}{=} e^{-F} \mathsf{P}_g^h \tag{113}$$

with F as above.

Proof. By the conformal quasi-invariance of the LQG measure as stated in (111) and by the conformal invariance of the bilinear form $\mathfrak{E}_g$,

$$\int_M \mathsf{P}_g^h u\, v\, d\mu_g^{\gamma h} = \mathfrak{E}_g^h(u, v) = \mathfrak{E}_{g'}^h(u, v) = \int_M \mathsf{P}_{g'}^h u\, v\, d\mu_{g'}^{\gamma h} = \int_M e^{Z^h} \mathsf{P}_{g'}^h u\, v\, d\mu_g^{\gamma h}$$

for all u and v in appropriate domains. Hence, $\mathsf{P}_g^h u = e^{Z^h} \mathsf{P}_{g'}^h u$. This proves the claim. □

Remark 5.18. The above construction can also be carried out with $\bar{\mu}_g^{\gamma h}$ instead of $\mu_g^{\gamma h}$ yielding the *adjusted random co-polyharmonic operator* $\bar{\mathsf{P}}_g^h$. In that case, we get for the conformal quasi-invariance the following formula:

$$\bar{\mathsf{P}}_{g'}^h \overset{(\mathrm{d})}{=} e^{-\bar{F}} \bar{\mathsf{P}}_g^h ,$$

where $\bar{F} = \gamma \left\langle h \,|\, e^{n\bar{\varphi}} \right\rangle + (n + \gamma^2/2)\varphi$.

6 | THE POLYAKOV–LIOUVILLE MEASURE

Our last objective in this paper is to propose a version of conformal field theory on compact manifolds of arbitrary even dimension, an approach based on Branson's Q-curvature. We provide a rigorous meaning to the Polyakov–Liouville measure $\boldsymbol{\pi}_g$, informally given as

$$\exp\left(-S_g(h)\right) dh$$

with the (nonexisting) uniform distribution dh on the set of fields (thought as sections of some bundles over M), and the action

$$S_g(h) := \int_M \left(\frac{1}{2} \left| \sqrt{\mathsf{p}_g}\, h \right|^2 + \Theta\, Q_g h + \frac{\Theta^*}{\mathsf{vol}_g(M)} h + m e^{\gamma h} \right) d\mathsf{vol}_g \ , \tag{114}$$

where $\mathsf{p}_g = a_n \mathsf{P}_g$ is the normalized co-polyharmonic operator (with the constant a_n from (32)), Q_g denotes Branson's curvature, and $m, \Theta, \Theta^*, \gamma$ are parameters — subjected to some restrictions specified below, in particular, $0 < |\gamma| < \sqrt{2n}$.

We note that the factor $\Theta^*/\mathsf{vol}_g(M)$ does not appear elsewhere in the literature. Its presence is needed to address constructions of *plain* objects, see §6.2, while the factor is not present when addressing adjusted objects, see §6.3.

6.1 | Heuristics and motivations

Before going into the details of our approach, let us briefly recall the longstanding challenge of conformal field theory and some recent breakthroughs in the two-dimensional case. Here, (114) becomes the celebrated Polyakov–Liouville action

$$S_g(h) = \int_M \left(\frac{1}{4\pi} |\nabla h|^2 + \frac{\Theta}{2} R_g\, h + \frac{\Theta^*}{\mathsf{vol}_g(M)} h + m\, e^{\gamma h} \right) d\mathsf{vol}_g \ , \tag{115}$$

where R_g is the scalar curvature and $m, \Theta, \Theta^*, \gamma$ are parameters. (Instead of m and Θ mostly in the literature, $\bar{\mu}$ and Q are used. However, in this paper, the latter symbols are already reserved for the LQG measure and Branson's curvature.) With the Polyakov–Liouville action, this ansatz for the measure $\boldsymbol{\pi}_g(dh) = \frac{1}{Z_g^*} e^{-S_g(h)} dh$ reflects the coupling of the gravitational field with a matter field. It can be regarded as quantization of the classical Einstein–Hilbert action $S_g^{EH}(h) = \frac{1}{2\kappa} \int_M (R_g - 2\Lambda)\, dx$ or, more precisely, of its coupling with a matter field

$$S_g^{EH}(h) = \int_M \left[\frac{1}{2\kappa} \big(R_g - 2\Lambda \big) + \mathcal{L}_M \right] dx \ .$$

In the case $n = 2$ and $Q^* = 0$, based on the concepts of Gaussian free fields and LQG measures, the rigorous construction of such a Polyakov–Liouville measure $\boldsymbol{\pi}_g$ has been carried out recently in [28] for surfaces of genus 0, [33] for surfaces of genus 1 (see also [50] for the disk), and in [47] for surfaces of higher genus. For related constructions, see [31]. The approach of [47] gives a rigorous meaning to

$$\boldsymbol{\pi}_g(dh) = \exp\left[- \int \left(\frac{\Theta}{2} R_g\, h + m\, e^{\gamma h} \right) d\mathsf{vol}_g \right] \exp\left(-\frac{1}{4\pi} \|\nabla h\|^2_{L^2_g} \right) dh$$

by setting

$$\boldsymbol{\nu}_g^*(dh) := \exp\left(-\frac{\Theta}{2} \big\langle h \,\big|\, R_g \big\rangle_g - m\, \bar{\mu}_g^{\gamma h}(M) \right) \hat{\boldsymbol{\nu}}_g(dh) \ ,$$

where $\bar{\mu}_g^{\gamma h}$ denotes the adjusted LQG measure on M with parameter $\gamma \in (0,2)$, and

$$\boldsymbol{\pi}_g(dh) := \sqrt{\frac{\mathrm{vol}_g(M)}{\det'(-\frac{1}{4\pi^2}\Delta_g)}}\, \boldsymbol{\nu}_g^*(dh)\ .$$

This provides a complete solution to the above-mentioned challenge in dimension 2.

In dimension greater than 2, the same constructions are considered on spheres by B. Cerclé in [16], and are anticipated in the physics literature by T. Levy and Y. Oz in [60]. Here, we carry out the same program with mathematical rigor and in full generality on arbitrary even-dimensional admissible manifolds.

Remark 6.1. The previous argumentation is based on the interpretation of $\widehat{\boldsymbol{\nu}}_g := \widehat{\mathsf{GFF}}_g$, the law of the ungrounded Gaussian free field, as a rigorous definition for the measure

$$\sqrt{\frac{\det'(-\frac{1}{4\pi^2}\Delta_g)}{\mathrm{vol}_g(M)}}\ \exp\Big(-\tfrac{1}{4\pi}\langle h| -\Delta h\rangle\Big)dh\ .$$

To justify the latter, recall that for a nonnegative symmetric operator A on a finite-dimensional Hilbert space H with orthonormal eigenbasis $\{e_1,\dots,e_n\}$ and eigenvalues $\{\lambda_1,\dots,\lambda_n\}$, we put

$$\int_H f(h)\exp\left(-\pi\,\langle h, Ah\rangle\right)dh := \int_{\mathbb{R}^n} f\bigg(\sum_i x_i e_i\bigg)\exp\bigg(-\pi\,\sum_i \lambda_i x_i^2\bigg)dx$$

and thus, in particular,

$$\int_H \exp\left(-\pi\,\langle h, Ah\rangle\right)dh = \frac{1}{\sqrt{\det(A)}}.$$

Therefore, for a positive self-adjoint operator A with discrete spectrum $\{\lambda_j\}_{j\in\mathbb{N}}$ on a Hilbert space H in analogy, we put

$$\int_H \exp\left(-\pi\,\langle h, Ah\rangle\right)dh =: \frac{1}{\sqrt{\det'(A)}},$$

where $\det'(A) := \exp(-\zeta_A'(0))$ denotes the *regularized determinant*, defined in terms of the meromorphic continuation of the function

$$\zeta_A(s) := \sum_{j\geqslant 1}\lambda_j^{-s},$$

initially defined for s with large enough real part. For $A = -\frac{1}{4\pi^2}\Delta$, this leads to the identification

$$\exp\Big(-\tfrac{1}{4\pi^2}\langle h| -\Delta h\rangle\Big)dh = \frac{1}{\sqrt{\det'(-\frac{1}{4\pi^2}\Delta)}}\,\boldsymbol{\nu}_g(dh) \qquad \text{on } \mathring{H}_g^{-\varepsilon}.$$

Furthermore, the orthogonal decomposition $\hbar = h + a\,e_0$ of $\hbar \in H_g^{-\varepsilon}$ into $h \in \mathring{H}_g^{-\varepsilon}$ and a multiple of the normalized eigenfunction $e_0 := \frac{1}{\sqrt{\mathrm{vol}_g(M)}}$ for the (single) eigenvalue 0 of the Laplacian leads to

$$\int_{H_g^{-\varepsilon}} f(\hbar)\exp\Big(-\tfrac{1}{4\pi^2}\langle \hbar| -\Delta \hbar\rangle\Big) d\hbar = \int_{\mathbb{R}}\left(\int_{\mathring{H}_g^{-\varepsilon}} f(h + a\, e_0)\exp\Big(-\tfrac{1}{4\pi^2}\langle h - \Delta h\rangle\Big) dh\right) da$$

$$= \sqrt{\mathsf{vol}_g(M)}\int_{\mathbb{R}}\left(\int_{\mathring{H}_g^{-\varepsilon}} \frac{f(h+a)}{\sqrt{\det'(-\frac{1}{4\pi^2}\Delta)}}\,\boldsymbol{\nu}_g(dh)\right) da$$

$$= \frac{\sqrt{\mathsf{vol}_g(M)}}{\sqrt{\det'(-\frac{1}{4\pi^2}\Delta)}}\int_{H_g^{-\varepsilon}} f(\hbar)\, d\widehat{\boldsymbol{\nu}}_g(d\hbar)$$

for any $f : H_g^{-\varepsilon} \to \mathbb{R}_+$. This yields the identification

$$\exp\Big(-\tfrac{1}{4\pi^2}\langle \hbar| -\Delta \hbar\rangle\Big) d\hbar = \frac{\sqrt{\mathsf{vol}_g(M)}}{\sqrt{\det'(-\frac{1}{4\pi^2}\Delta)}}\,\widehat{\boldsymbol{\nu}}_g(d\hbar) \qquad \text{on } H_g^{-\varepsilon}.$$

A remarkable property of the Polyakov–Liouville action is that it quantifies the conformal quasi-invariance of the functional determinant in dimension 2. Namely, we have that [67, Eq. (1.13)]:

$$\log\frac{\det'(-\frac{1}{4\pi^2}\Delta_{g'})}{\mathsf{vol}_{g'}(M)} - \log\frac{\det'(-\frac{1}{4\pi^2}\Delta_g)}{\mathsf{vol}_g(M)} = -\frac{1}{12\pi}\int 2\varphi\,\mathsf{scal}_g + |\nabla\varphi|_g^2\, d\mathsf{vol}_g. \tag{116}$$

Thus, we can see the Polyakov–Liouville action as a potential accounting for the variation of the functional determinant of the Laplacian coupled with the volume. It is conjectured (see [10, Equ. (6)], [25, Equ. (5.9b)] and the references therein) that a physically relevant Polyakov formula for $n > 2$ should involve the (normalized) co-polyharmonic operators. Under our admissibility, it should take the form:

$$\log\det'\mathsf{p}_g - \log\det'\mathsf{p}_{g'} = \Theta\int\Big[\frac{1}{2}\varphi\,\mathsf{P}_g\varphi + \varphi\, Q_g\Big]\, \mathrm{d}\,\mathsf{vol}_g + \int F_{g'}\, d\mathsf{vol}_{g'} - \int F_g\, d\mathsf{vol}_g + G\ ,$$

where Θ is a constant, F_g and $F_{g'}$ are local scalar invariants, and G is a global term. The first integral on the right-hand side here is regarded as the "universal part." In view of this formula, let us define a higher dimensional equivalent of the $2d$ Polyakov–Liouville action:

$$S_g(h) = \Theta\int h\,Q_g\, d\mathsf{vol}_g + \frac{\Theta^*}{\mathsf{vol}_g(M)}\int h\, d\mathsf{vol}_g + m\int e^{\gamma h} d\mathsf{vol}_g + \frac{1}{2}\mathfrak{p}_g(h,h)\ .$$

Remark 6.2. Minimizers of S_g satisfy

$$\mathsf{p}_g h + \Theta Q_g + \frac{\Theta^*}{\mathsf{vol}_g(M)} + m\gamma e^{\gamma h} = 0.$$

If we choose $\Theta^* = 0$, $\Theta = \frac{na_n}{\gamma}$, $m = -\frac{na_n}{\gamma^2}\bar{Q}$ for some $\bar{Q} \in \mathbb{R}$ and put $\varphi = \frac{\gamma}{n}h$, then this reads as

$$\frac{1}{a_n}\,\mathsf{p}_g\varphi + Q_g = e^{n\varphi}\bar{Q}.$$

In other words, $g' = e^{2\varphi}g$ is a metric of constant Branson curvature $Q_{g'} = \bar{Q}$.

The remainder of this section is devoted to give a rigorous meaning to the measure

$$\boldsymbol{\pi}_g(dh) = \exp(-S_g(h))dh.$$

As an ansatz, we regard the quantity $\frac{1}{Z_g}\exp(-\frac{1}{2}\mathfrak{p}_g(h,h))dh$ as an informal definition of the law of the ungrounded co-polyharmonic field. With this interpretation, we regard $\int e^{\gamma h}\mathsf{dvol}_g$ as the volume of M with respect to the LQG measure. Since the latter comes in two versions —the plain and the adjusted Liouville measure — we obtain two conformally quasi-invariant rigorous definitions of the above measure, denoted henceforth by $\boldsymbol{\nu}^*_g$ and $\bar{\boldsymbol{\nu}}^*_g$.

Remark 6.3. Before going into further details, let us have a naive look on the transformation property of our action functional under conformal changes that would apply if the random field h were smooth. Choose $\Theta^* = 0$. Then, by a direct computation, we have that for all φ smooth and all $h \in H^{n/2}_g$:

$$S_{e^{2\varphi}g}\left(h - \frac{n}{\gamma}\varphi\right) = S_g(h) + \left(\frac{\Theta}{a_n} - \frac{n}{\gamma}\right)\mathfrak{p}_g(h,\varphi) + \left(\frac{n^2}{2\gamma^2} - \frac{\Theta}{a_n}\frac{n}{\gamma}\right)\mathfrak{p}_g(\varphi,\varphi) - \Theta\frac{n}{\gamma}\int \varphi\, Q_g\, \mathsf{dvol}_g\ ,$$

where we used that $Q_{e^{2\varphi}g} = e^{-n\varphi}(Q_g + \frac{1}{a_n}\mathsf{p}_g\varphi)$. In particular, when selecting the special value $\Theta = a_n\frac{n}{\gamma}$ the above expression simplifies to

$$S_{e^{2\varphi}g}\left(h - \frac{n}{\gamma}\varphi\right) = S_g(h) - \frac{n^2}{\gamma^2}\left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n\int \varphi\, Q_g\, \mathsf{dvol}_g\right].$$

Therefore, writing T for the shift by $h \mapsto h + \frac{n}{\gamma}\varphi$, we expect the following quasi-conformal invariance:

$$\log\frac{dT_*\boldsymbol{\pi}_{e^{2\varphi}g}}{d\boldsymbol{\pi}_g}(h) = \frac{n^2}{\gamma^2}\left(\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n\int \varphi\, Q_g\, \mathsf{dvol}_g\right).$$

However, due to the choice of the measure $\boldsymbol{\nu}_*$ the random field h will not be smooth. As a consequence, the quasi-conformal invariance arises at a different value of Θ (and/or Θ^*). Indeed, the renormalization of the adjusted (or plain) LQG measure $\bar{\mu}^{\gamma h}_g$ (or $\mu^{\gamma h}_g$, resp.) produces in an additional term which corresponds to quasi-invariance under the shift

$$h \mapsto h + \left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)\varphi$$

(or $h \mapsto h + \frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar{\varphi}$ for some function $\bar{\varphi}$ given in terms of φ). We derive rigorous statements below. For convenience, we treat the two procedures –in spite of their similarity– for the "plain" and "adjusted" cases separately.

Remark 6.4. The approach involving the adjusted measure $\bar{\boldsymbol{\nu}}^*_g$ is similar to (and inspired by) that of [47] in the case $n = 2$. Results concerning the plain measure $\boldsymbol{\nu}^*_g$ seem to be new even in the two-dimensional case.

6.2 | The plain Polyakov–Liouville measure

Let us address the challenge of giving a rigorous meaning to

$$d\boldsymbol{\pi}_g(h) = \exp\Big(-\int \big(\Theta\, Q_g h + \Theta^*\langle h\rangle_g + m e^{\gamma h}\big) d\mathrm{vol}_g\Big) \exp\Big(-\frac{1}{2}\mathfrak{p}_g(h,h)\Big) dh$$

on admissible manifolds of arbitrary even dimension. Assume for the sequel that $|\gamma| < \sqrt{2n}$, and let

$$\boldsymbol{\nu}_g := \mathsf{CGF}_g$$

denote the law of the co-polyharmonic field, a (rigorously defined) probability measure on $\mathring{H}^{-\varepsilon}_g$ for some/any $\varepsilon > 0$. For convenience, we choose $\varepsilon = n/2$. Furthermore, let

$$\widehat{\boldsymbol{\nu}}_g := \widehat{\mathsf{CGF}}_g \tag{117}$$

denote the (infinite) measure on $H^{-n/2}_g$ introduced in Proposition 3.16 as the distribution of the ungrounded co-polyharmonic field on (M, g). As outlined in Section 3 and Remark 6.1, the latter admits a heuristic characterization as

$$d\widehat{\boldsymbol{\nu}}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{p}_g(h,h)\Big) dh$$

with

$$Z_g = \sqrt{\frac{\mathrm{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)}}.$$

Proceeding as in the two-dimensional case, in terms of this measure, we define the measure

$$d\boldsymbol{\nu}^*_g(h) := \exp\Big(-\Theta\big\langle h \,\big|\, Q_g\big\rangle_g - \Theta^*\langle h\rangle_g - m\,\mu^{\gamma h}_g(M)\Big)\, d\widehat{\boldsymbol{\nu}}_g(h) \tag{118}$$

on $H^{-n/2}_g$ with associated *partition function*

$$Z^*_g := \int_{H^{-n/2}} d\,\boldsymbol{\nu}^*_g(h) \in (0,\infty]\,,$$

where $\Theta, \Theta^*, m, \gamma \in \mathbb{R}$ are parameters with $m > 0$, $0 < |\gamma| < \sqrt{2n}$, and where $\mu^{\gamma h}_g$ denotes the plain LQG measure on the n-dimensional manifold M. Moreover, $\big\langle h \,\big|\, Q_g\big\rangle_g$ denotes the pairing between the random field h and the (scalar valued) Q-curvature, and $\langle h\rangle_g = \frac{1}{\mathsf{vol}_g(M)}\langle h\,|\,\mathbf{1}\rangle$ denotes the pairing between h and the constant function $\mathbf{1}$, normalized by the volume of M. Set $Q(M) := Q(M,g)$.

Theorem 6.5. *Assume that $0<\gamma<\sqrt{2n}$ and $\Theta\, Q(M)+\Theta^*<0$. Then, $\boldsymbol{\nu}^*_g$ is a finite measure and so is*

$$\boldsymbol{\pi}_g := \sqrt{\frac{\mathsf{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)}}\cdot \boldsymbol{\nu}^*_g .$$

The normalizations of them coincide and provide a well-defined probability measure on $H_g^{-n/2}$,

$$\boldsymbol{\nu}^\sharp_g := \frac{1}{Z^*_g}\boldsymbol{\nu}^*_g = \frac{1}{Z^\pi_g}\boldsymbol{\pi}_g$$

with $Z^\pi_g := \int_{H^{-n/2}} d\,\boldsymbol{\pi}_g(h)$.

Proof.

$$\begin{aligned}
Z^*_g &= \int_{H^{-n/2}} \exp\Big(-\Theta\big\langle h\,\big|\,Q_g\big\rangle_g - \Theta^*\langle h\rangle_g - m\,\mu_g^{\gamma h}(M)\Big)\,d\widehat{\boldsymbol{\nu}}_g(h)\\
&= \int_{H^{-n/2}}\int_{\mathbb{R}} \exp\Big(-\Theta\big\langle h\,\big|\,Q_g\big\rangle_g - a(\Theta Q(M)+\Theta^*) - m\,e^{\gamma a}\mu_g^{\gamma h}(M)\Big)\,da\,d\boldsymbol{\nu}_g(h)\\
&\overset{(a)}{=} \int_{H^{-n/2}} e^{-\Theta\left\langle h\,\middle|\,Q_g\right\rangle_g}\int_0^\infty \left(\frac{t}{m\,\mu_g^{\gamma h}(M)}\right)^{-\frac{\Theta\,Q(M)+\Theta^*}{\gamma}} e^{-t}\,\frac{dt}{\gamma t}\,d\boldsymbol{\nu}_g(h)\\
&\overset{(b)}{=} \frac{1}{\gamma}\,\Gamma\Big(-\frac{\Theta\,Q(M)+\Theta^*}{\gamma}\Big)\cdot\int_{H^{-n/2}} e^{-\Theta\left\langle h\,\middle|\,Q_g\right\rangle_g}\Big(m\,\mu_g^{\gamma h}(M)\Big)^{\frac{\Theta\,Q(M)+\Theta^*}{\gamma}}\,d\boldsymbol{\nu}_g(h)\;.
\end{aligned}$$

Here, (a) follows by change of variables $a\mapsto t := m\,e^{\gamma a}\mu^h_{g,\gamma}(M)$, and (b) by the very definition of Euler's Γ function. The final integral then can be estimated according to

$$\begin{aligned}
&\int_{H^{-n/2}} e^{-\Theta\left\langle h\,\middle|\,Q_g\right\rangle_g}\Big(m\,\mu_g^{\gamma h}(M)\Big)^{\frac{\Theta\,Q(M)+\Theta^*}{\gamma}}\,d\boldsymbol{\nu}_g(h)\\
&\quad\leqslant \left(\int_{H^{-n/2}} e^{-2\Theta\left\langle h\,\middle|\,Q_g\right\rangle_g}\,d\boldsymbol{\nu}_g(h)\right)^{1/2}\cdot\left(\int_{H^{-n/2}}\Big(m\,\mu_g^{\gamma h}(M)\Big)^{\frac{2(\Theta\,Q(M)+\Theta^*)}{\gamma}}\,d\boldsymbol{\nu}_g(h)\right)^{1/2}.
\end{aligned}$$

The finiteness of the first term on the right-hand side is obvious by the defining property of $\boldsymbol{\nu}_g$:

$$\int_{H^{-n/2}} e^{-2\Theta\left\langle h\,\middle|\,Q_g\right\rangle_g}\,d\boldsymbol{\nu}_g(h) = e^{2\Theta^2\,\mathcal{K}_g(Q_g,Q_g)}\;.$$

The finiteness of $\int_{H^{-n/2}} \mu_g^{\gamma h}(M)^{\frac{2(\Theta\,Q(M)+\Theta^*)}{\gamma}}\,d\boldsymbol{\nu}_g(h)$ for $\frac{\Theta\,Q(M)+\Theta^*}{\gamma}<0$ follows from Theorem 4.1 (iii). ☐

Remark 6.6. Assuming that Θ is positive, the finiteness assumption $\Theta\, Q(M) + \Theta^* < 0$ in the above theorem is equivalent to saying that the constant $-\Theta^*/\Theta$ is larger than the total Q-curvature.

Theorem 6.7. *Assume that $0 < \gamma < \sqrt{2n}$, $\Theta = a_n \frac{n}{\gamma}$, and $\Theta^* = \gamma$. Then, $\boldsymbol{\nu}^*_g$ is conformally quasi-invariant modulo shift in the sense that for all $\varphi \in C^\infty(M)$ and $g' = e^{2\varphi} g$,*

$$\boldsymbol{\nu}^*_{g'} = Z(g,\varphi) \cdot T_* \boldsymbol{\nu}^*_g \ , \tag{119}$$

where T_* denotes the push forward under the shift $T : h \mapsto h - \frac{n}{\gamma}\varphi - \frac{\gamma}{2}\bar{\varphi}$ on $H^{-n/2}_g$ with $\bar{\varphi}$ defined as in (85). The A-type conformal anomaly *$Z(g,\varphi)$ is given as*

$$Z(g,\varphi) := \exp\left[\Theta \int \left(\frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar{\varphi}\right) Q_g \mathsf{dvol}_g + n\langle\varphi\rangle_{g'} + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi)\right] . \tag{120}$$

In particular, $Z(g,\varphi) = Z^*_{e^{2\varphi}g}/Z^*_g$ provided $Z^*_g < \infty$.

Proof. For the sake of brevity, let us write

$$S_g(h) = \Theta\big\langle h \,\big|\, Q_g \big\rangle_g + \Theta^* \langle h\rangle_g + m\mu^{\gamma h}_g(M) \ .$$

We also set $\Phi := (\frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar{\varphi}) \in C^\infty(M)$. Let $F : H^{-n/2}_g \to \mathbb{R}_+$ be measurable. Then, by Girsanov theorem (Corollary 3.18) for $\hat{\boldsymbol{\nu}}_{g'}$, we find that

$$\int_{H^{-n/2}} F d\boldsymbol{\nu}^*_{g'} = \int F(h-\Phi) \exp\big(-S_{g'}(h-\Phi)\big) \cdot \exp\Big(\langle\!\langle h | \mathsf{p}_{g'}\Phi \rangle\!\rangle_{g'} - \frac{1}{2}\mathfrak{p}_{g'}(\Phi,\Phi)\Big) d\hat{\boldsymbol{\nu}}_{g'}(h) \ .$$

By Corollary 4.17 and Theorem 4.1(i), we have that

$$\begin{aligned}\int_{H^{-n/2}} F d\boldsymbol{\nu}^*_{g'} = \int F(h-\Phi) &\exp\Big(-\Theta\big\langle h-\Phi \,\big|\, Q_{g'}\big\rangle_{g'} - m\mu^{\gamma h}_g(M)\Big) \\ &\cdot \exp\Big(-\Theta^*\big\langle h-\Phi \,\big|\, \mathbf{1}\big\rangle_{g'}\Big) \\ &\cdot \exp\Big(\big\langle h \,\big|\, \mathsf{p}_{g'}\Phi\big\rangle_{g'} - \frac{1}{2}\mathfrak{p}_{g'}(\Phi,\Phi)\Big) d\hat{\boldsymbol{\nu}}_g(h) \ .\end{aligned}$$

Now recall that $\mathsf{p}_{g'} u = e^{-n\varphi}\mathsf{p}_g u$, that $\mathfrak{p}_g$ is conformally invariant, and that, by Proposition 21, $Q_{g'} = e^{-n\varphi}(Q_g + \frac{1}{a_n}\mathsf{p}_g\varphi)$. Thus, we obtain

$$\begin{aligned}\int_{H^{-n/2}} F d\boldsymbol{\nu}^*_{g'} = \int F(h-\Phi) &\exp\Big(-\Theta\big\langle h-\Phi \,\big|\, Q_g\big\rangle_g - m\mu^{\gamma h}_g(M)\Big) \\ &\cdot \exp\Big(-\Theta^*\big\langle h-\Phi \,\big|\, \mathbf{1}\big\rangle_{g'}\Big) \\ &\cdot \exp\Big(-\frac{\Theta}{a_n}\big\langle h-\Phi \,\big|\, \mathsf{p}_g\varphi\big\rangle_g + \big\langle h \,\big|\, \mathsf{p}_g\Phi\big\rangle_g - \frac{1}{2}\mathfrak{p}_g(\Phi,\Phi)\Big) d\hat{\boldsymbol{\nu}}_g(h) \ .\end{aligned}$$

Since we have chosen $\Theta = a_n \frac{n}{\gamma}$, some of the terms in the last line cancel out and we get:

$$\int_{H^{-n/2}} F d\boldsymbol{\nu}^*_{g'} = \int F(h-\Phi) \exp\left(-\Theta\left\langle h-\Phi \,\middle|\, Q_g\right\rangle_g - m\mu_g^{\gamma h}(M)\right)$$
$$\exp\left(-\Theta^*\left\langle h-\Phi \,\middle|\, \mathbf{1}\right\rangle_{g'}\right)$$
$$\exp\left(\frac{\gamma}{2}\left\langle h \,\middle|\, \mathsf{p}_g\bar\varphi\right\rangle_g + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi) - \frac{\gamma^2}{8}\mathfrak{p}_g(\bar\varphi,\bar\varphi)\right) d\hat{\boldsymbol{\nu}}_g(h)\ .$$

Now by definition of $\bar\varphi$, we get

$$\mathsf{p}_g\bar\varphi = \frac{2}{\mathsf{vol}_{g'}(M)}\pi_g(e^{n\varphi}) = \frac{2}{\mathsf{vol}_{g'}(M)}e^{n\varphi} - \frac{2}{\mathsf{vol}_g(M)}\ .$$

In particular, for every $h \in H_g^{-n/2}$,

$$\left\langle h \,\middle|\, \mathsf{p}_g\bar\varphi\right\rangle = 2\langle h\rangle_{g'} - 2\langle h\rangle_g\ . \tag{121}$$

As a consequence,

$$\int_{H^{-n/2}} F d\boldsymbol{\nu}^*_{g'} = \int F(h-\Phi) \exp\left(-\Theta\left\langle h \,\middle|\, Q_g\right\rangle_g + \Theta\int \Phi Q_g \mathsf{dvol}_g - m\mu_g^{\gamma h}(M)\right)$$
$$\exp\left(-\Theta^*\langle h\rangle_{g'} + \Theta^*\langle\Phi\rangle_{g'}\right)$$
$$\exp\left(\gamma\left(\langle h\rangle_{g'} - \langle h\rangle_g\right) + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi) - \frac{\gamma^2}{8}\mathfrak{p}_g(\bar\varphi,\bar\varphi)\right) d\hat{\boldsymbol{\nu}}_g(h)\ .$$

Thus, the choice of $\Theta^* = \gamma$, after cancelations and rearrangement, yields

$$\int_{H^{-n/2}} F d\boldsymbol{\nu}^*_{g'} = \int F(h-\Phi) \exp\left(-\Theta\left\langle h \,\middle|\, Q_g\right\rangle_g - \Theta^*\langle h\rangle_g - m\mu_g^{\gamma h}(M)\right) d\hat{\boldsymbol{\nu}}_g(h)$$
$$\cdot \exp\left(\int \Phi\left(\Theta Q_g + \Theta^*\frac{e^{n\varphi}}{\mathsf{vol}_{g'}(M)}\right)\mathsf{dvol}_g + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi) - \frac{\gamma^2}{8}\mathfrak{p}_g(\bar\varphi,\bar\varphi)\right)\ . \tag{122}$$

Again, in light of (121), we further have that

$$\frac{1}{2}\mathfrak{p}_g(\bar\varphi,\bar\varphi) = \langle\bar\varphi\rangle_{g'} - \langle\bar\varphi\rangle_g\ .$$

Substituting the definitions of $\Theta^* := \gamma$ and $\Phi := (\frac{n}{\gamma}\varphi + \frac{\gamma}{2}\bar\varphi)$ then yields

$$\int \Phi\left(\Theta Q_g + \Theta^*\frac{e^{n\varphi}}{\mathsf{vol}_{g'}(M)}\right)\mathsf{dvol}_g + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi) - \frac{\gamma^2}{8}\mathfrak{p}_g(\bar\varphi,\bar\varphi) =$$
$$= \Theta\int \Phi Q_g \mathsf{dvol}_g + n\langle\varphi\rangle_{g'} + \frac{\gamma^2}{4}\left(\langle\bar\varphi\rangle_{g'} + \langle\bar\varphi\rangle_g\right) + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi)\ .$$

Finally, by definition (85) of $\bar\varphi$, and since $\mathsf{k}_g(e^{n\varphi}) \in \mathring{L}^2_g$,

$$\begin{aligned}\langle\bar\varphi\rangle_{g'} + \langle\bar\varphi\rangle_g &= \frac{2}{\mathrm{vol}_{g'}(M)^2}\int_M e^{n\varphi}\mathsf{k}_g(e^{n\varphi})d\mathrm{vol}_g - \frac{1}{\mathrm{vol}_{g'}(M)^2}\mathcal{K}_g(e^{n\varphi}, e^{n\varphi}) \\ &\quad + \frac{2}{\mathrm{vol}_{g'}(M)\mathrm{vol}_g(M)}\int_M \mathsf{k}_g(e^{n\varphi})d\mathrm{vol}_g - \frac{1}{\mathrm{vol}_{g'}(M)^2}\mathcal{K}_g(e^{n\varphi}, e^{n\varphi}) \\ &= \frac{2}{\mathrm{vol}_{g'}(M)^2}\mathcal{K}_g(e^{n\varphi}, e^{n\varphi}) - \frac{1}{\mathrm{vol}_{g'}(M)^2}\mathcal{K}_g(e^{n\varphi}, e^{n\varphi}) \\ &\quad + 0 - \frac{1}{\mathrm{vol}_{g'}(M)^2}\mathcal{K}_g(e^{n\varphi}, e^{n\varphi}) \\ &= 0\ ,\end{aligned}$$

and therefore,

$$\begin{aligned}&\int \Phi\left(\Theta Q_g + \Theta^*\frac{e^{n\varphi}}{\mathrm{vol}_{g'}(M)}\right)d\mathrm{vol}_g + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi) - \frac{\gamma^2}{8}\mathfrak{p}_g(\bar\varphi,\bar\varphi) = \\ &= \Theta\int \Phi Q_g d\mathrm{vol}_g + n\langle\varphi\rangle_{g'} + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi)\ .\end{aligned} \tag{123}$$

Substituting (123) into (122), we finally have that

$$\begin{aligned}\int_{H^{-n/2}} F d\boldsymbol{\nu}^*_{g'} &= \int F(h-\Phi)\exp\left(-\Theta\left\langle h \,\middle|\, Q_g\right\rangle_g - \Theta^*\langle h\rangle_g - m\mu^{\gamma h}_g(M)\right)d\hat{\boldsymbol{\nu}}_g(h) \\ &\quad\cdot\exp\left(\Theta\int \Phi Q_g d\mathrm{vol}_g + n\langle\varphi\rangle_{g'} + \frac{n^2}{2\gamma^2}\mathfrak{p}_g(\varphi,\varphi)\right)\ .\end{aligned}$$

This concludes the proof of the conformal quasi-invariance. To conclude for the expression of $Z(g,\varphi)$, we take $F = \mathbf{1}$. □

Corollary 6.8. *Assume that $\Theta = a_n\frac{n}{\gamma}$, $\Theta^* = \gamma$, and $\gamma^2 < -n\, a_n\, Q(M)$. Then $Z(g,\varphi) = \frac{Z^*_{g'}}{Z^*_g}$, and $\boldsymbol{\nu}^\sharp_g$ is conformally invariant modulo shift:*

$$\boldsymbol{\nu}^\sharp_{e^{2\varphi}g} = T_*\boldsymbol{\nu}^\sharp_g \tag{124}$$

with $T : h \mapsto h - n\varphi/\gamma - \gamma\bar\varphi/2$.

Proof. We have

$$\boldsymbol{\nu}^\sharp_{g'} = \frac{\boldsymbol{\nu}^*_{g'}}{Z^*_{g'}} = \frac{Z(g,\varphi)}{Z^*_{g'}}\cdot T_*\boldsymbol{\nu}^*_g = \frac{Z^*_g}{Z^*_{g'}}Z(g,\varphi)\cdot T_*\boldsymbol{\nu}^\sharp = T_*\boldsymbol{\nu}^\sharp\ .$$

□

Remark 6.9. With the choices $\Theta := \frac{na_n}{\gamma}$ and $\Theta^* := \gamma$ from above, the condition $\Theta\, Q(M) + \Theta^* < 0$ reads as $\frac{a_n n}{\gamma^2}Q(M) + 1 < 0$ or, in other words,

$$\gamma^2 < -n\, a_n\, Q(M)\ . \tag{125}$$

Remark 6.10. In view of Corollaries 3.17 and 4.16, the quasi-invariance assertion in the previous Theorem 6.7 and Corollary 6.8 also holds under the more general class of conformal transformations in the sense of Definition 1.1(ii). In particular, the plain Polyakov–Liouville measure is quasi-invariant under isometric transformations $\Phi : M \to M'$.

6.3 | The adjusted Polyakov–Liouville measures

As anticipated, our results for the plain Polyakov–Liouville measure can also be recast in the setting of the adjusted Polyakov–Liouville measure. Let us set

$$d\bar{\boldsymbol{\nu}}^*_g(h) = \exp\Big(-\Theta\big\langle h \,\big|\, Q_g\big\rangle_g - m\bar{\mu}^{\gamma h}_g(M)\Big) d\widehat{\boldsymbol{\nu}}_g(h) \ ,$$

which corresponds to the *adjusted Polyakov–Liouville measure*. The associated *partition function* is

$$\bar{Z}^*_g := \int_{H^{-n/2}} d\,\bar{\boldsymbol{\nu}}^*_g(h) \ .$$

As for the plain measure, we have the following result.

Theorem 6.11. *Assume that* $0 < \gamma < \sqrt{2n}$ *and* $\Theta\, Q(M) < 0$. *Then,* $\bar{\boldsymbol{\nu}}^*_g$ *is a finite measure and so is*

$$\bar{\boldsymbol{\pi}}_g := \sqrt{\frac{\mathrm{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)}} \cdot \bar{\boldsymbol{\nu}}^*_g.$$

The measure $\bar{\boldsymbol{\pi}}_g$ is in fact the "standard" Polyakov–Liouville measure considered in the two-dimensional case; see, for example, [28, §3.2] and [47, Prop. 4.1]

Proof. The proof is the same as in Theorem 6.5, simply remarking that Theorem 4.1 (iii) also applies for $\bar{\mu}^h$ instead of μ^h. □

The next result extends to admissible manifolds the same assertion for spheres in [16, Thm. 3.10].

Theorem 6.12. *Assume that* $0 < \gamma < \sqrt{2n}$ *and that* $\Theta = a_n(\frac{n}{\gamma} + \frac{\gamma}{2})$. *Let* φ *be smooth and* $g' = e^{2\varphi} g$. *Then,* $\bar{\boldsymbol{\nu}}^*$ *is conformally quasi-invariant under the shift* $T : h \mapsto h - (\frac{n}{\gamma} + \frac{\gamma}{2})\varphi$, *namely,*

$$\bar{\boldsymbol{\nu}}^*_{g'} = \bar{Z}(g,\varphi) \cdot T_* \bar{\boldsymbol{\nu}}^*_g \ ,$$

where

$$\bar{Z}(g,\varphi) = \exp\left(\left(\frac{n}{\gamma} + \frac{\gamma}{2}\right)^2 \left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n \int \varphi\, Q_g\, d\mathrm{vol}_g\right]\right).$$

Proof. Let $F : H_g^{-n/2} \to \mathbb{R}_+$ be measurable. Write $\Phi = (\frac{n}{\gamma} + \frac{\gamma}{2})\varphi \in C^\infty(M)$. By Girsanov's theorem for $\hat{\boldsymbol{\nu}}_{g'}$ (Corollary 3.18), we have

$$\int_{H^{-n/2}} F d\bar{\boldsymbol{\nu}}^*_{g'} = \int F(h-\Phi) \exp\left(-\Theta\langle h-\Phi \,|\, Q_{g'}\rangle_{g'} - m\bar{\mu}^{\gamma(h-\Phi)}_{g'}(M)\right)$$
$$\cdot \exp\left(\left\langle h \,\middle|\, \mathsf{p}_{g'}\frac{\Theta}{a_n}\varphi\right\rangle_{g'} - \frac{1}{2}\mathfrak{p}_{g'}\left(\frac{\Theta}{a_n}\varphi, \frac{\Theta}{a_n}\varphi\right)\right) d\hat{\boldsymbol{\nu}}_{g'}(h)\,.$$

In view of Theorems 3.16 and 4.20, we thus get

$$\int_{H^{-n/2}} F d\bar{\boldsymbol{\nu}}^*_{g'} = \int F(h-\Phi) \exp\left(-\Theta\left\langle h-\Phi \,\middle|\, Q_g + \frac{1}{a_n}\mathsf{p}_g\varphi\right\rangle_g - m\bar{\mu}^{\gamma h}_g(M)\right)$$
$$\cdot \exp\left(\frac{\Theta}{a_n}\langle h \,|\, \mathsf{p}_g\varphi\rangle_g - \frac{\Theta^2}{2a_n^2}\mathfrak{p}_g(\varphi,\varphi)\right) d\hat{\boldsymbol{\nu}}_g(h)\,.$$

Expanding $\left\langle h - \Phi_g \,|\, Q_g + \frac{1}{a_n}\mathsf{p}_g\varphi\right\rangle_g$ cancels out with some term on the second line and we obtain the announced result. □

Corollary 6.13. *Assume $Q(M) < 0$ and set $\bar{\boldsymbol{\nu}}^\sharp_g := \frac{1}{\bar{Z}^*}\bar{\boldsymbol{\nu}}^*_g$. Then with Θ and T as above,*

$$\bar{\boldsymbol{\nu}}^\sharp_{g'} = T_*\bar{\boldsymbol{\nu}}^\sharp_g\,.$$

Remark 6.14. In dimension $n = 2$, the result of the previous theorem is consistent with the results in [47]. Indeed, the main result there (Thm. 1) deals with conformal changes of the Polyakov–Liouville measure

$$\bar{\boldsymbol{\pi}}_g := \sqrt{\frac{\mathsf{vol}_g(M)}{\det'(-\frac{1}{4\pi^2}\Delta_g)}} \cdot \bar{\boldsymbol{\nu}}^*_g.$$

In our notation and with Θ and T as in Theorem 6.12, the conformal anomaly for this measure in the case $n = 2$ is

$$\frac{d\bar{\boldsymbol{\pi}}_{g'}}{dT_*\bar{\boldsymbol{\pi}}_g} = \exp\left(\left[\frac{1}{6} + \left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)^2\right] \cdot \left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n\int \varphi\, Q_g \,\mathsf{dvol}_g\right]\right)\,. \tag{126}$$

The factor $\exp(\frac{1}{6}\left[\frac{1}{2}\mathfrak{p}_g(\varphi,\varphi) + a_n\int\varphi\, Q_g\,\mathsf{dvol}_g\right]$ here accounts for the conformal transformation of the prefactor $\sqrt{\frac{\mathsf{vol}_g(M)}{\det'(-\frac{1}{2\pi}\Delta_g)}}$ of the measure $\bar{\boldsymbol{\pi}}_g$. In dimension $n = 2$,

$$\log\det{}'(-t\Delta_g) = \log\det{}'(-\Delta_g) + \log(t)\left(1 - \frac{\chi(M)}{6}\right) \qquad \forall t > 0$$

with the latter term on the right-hand side here being conformally invariant, and the conformal change of

$$\log \frac{\mathrm{vol}_{g'}(M)}{\det'(-\Delta_{g'})} - \log \frac{\mathrm{vol}_g(M)}{\det'(-\Delta_g)}$$

is given explicitly in terms of the so-called *Polyakov formula* [67, Equ. (1.13)], cf. (116). The quantity $\mathsf{c} := 1 + 6(\frac{2}{\gamma} + \frac{\gamma}{2})^2$ appearing in (126) is the celebrated *central charge*, taking values in the interval $(25, \infty)$.

Indeed, with this choice of c, with $\omega = 2\varphi$, $a_2 = \frac{1}{2\pi}$, $K_g = 2Q_g$, and $\mathsf{p}_g(\omega, \omega) = \frac{1}{2\pi}\int |d\omega|_g^2 d\mathrm{vol}_g$, the right-hand side of (126) reads as

$$\exp\left(\frac{\mathsf{c}}{96\pi} \cdot \int \left[|d\omega|_g^2 + 2K_g\omega\right] d\mathrm{vol}_g\right) \tag{127}$$

exactly as in [47].

Corollary 6.15. *Assume that $n = 4$ and put similarly as before*

$$\bar{\boldsymbol{\pi}}_g := \sqrt{\frac{\mathrm{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)}} \cdot \bar{\boldsymbol{\nu}}_g^*.$$

Then with Θ and T as in Theorem 6.12,

$$\frac{d\bar{\boldsymbol{\pi}}_{g'}}{dT_*\bar{\boldsymbol{\pi}}_g} = \exp\left(\left[\frac{7}{45} + \left(\frac{4}{\gamma} + \frac{\gamma}{2}\right)^2\right] \cdot \left[\frac{1}{2}\mathfrak{p}_g(\varphi, \varphi) + a_n \int \varphi\, Q_g\, d\mathrm{vol}_g\right]\right)$$
$$\cdot \exp\left(\frac{1}{45\pi^2}\left[-\int \mathrm{scal}_{g'}^2 d\mathrm{vol}_{g'} + \int \mathrm{scal}_g^2 d\mathrm{vol}_g\right]\right) \cdot \exp\left(\frac{1}{1440\pi^2}\int \varphi |W|^2 d\mathrm{vol}_g\right),$$

where W denotes the Weyl tensor.

Proof. Let $\zeta_g(s) = \sum_{j\geqslant 1}(\nu_j/a_n)^{-s}$. Then $-\log\det'(\mathsf{P}_g) = \zeta_g'(0)$ and thus $-\log\det'(t\mathsf{P}_g) = \zeta_g'(0) - \log(t)\,\zeta_g(0)$ for every $t > 0$. The value

$$\zeta_g(0) = -1 + \int U_4(\mathsf{P}_g) d\mathrm{vol}_g$$

is a conformal invariant [9, Lemma 2]. Therefore,

$$\log \frac{\mathrm{vol}_{g'}(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_{g'})} - \log \frac{\mathrm{vol}_g(M)}{\det'(\frac{1}{2\pi}\mathsf{p}_g)} = \log \frac{\mathrm{vol}_{g'}(M)}{\det'(\mathsf{P}_{g'})} - \log \frac{\mathrm{vol}_g(M)}{\det'(\mathsf{P}_g)}.$$

The right-hand side is calculated explicitly in [9, Thm. 4]. Together with Theorem 6.12 above, this yields the claim. ☐

6.4 | Some examples

The above assertions impose two conditions on a given manifold (M, g): positivity of the co-polyharmonic operator P_g and negativity of the total Q-curvature $Q(M)$.

Let us present some examples of such manifolds.

Example 6.16 ($n = 2$). Every compact Riemannian surface of genus $\geqslant 2$ satisfies both of these conditions.

Example 6.17 ($n = 2, 6, 10, \ldots$). Every compact hyperbolic Riemannian manifold of dimension $n = 4\ell + 2$ for some $\ell \in \mathbb{N}$ and with $\lambda_1 > \frac{n(n-2)}{4}$ satisfies both of these conditions.

Proof. Combine Proposition 2.5 and Example 1.14. □

Example 6.18 ($n = 4$). Let $M = M_1 \times M_2$ where M_1 and M_2 are compact Riemannian surfaces of constant curvature k_1 and k_2, resp.

(i) Then $Q_g < 0$ if and only if

$$|k_1 + k_2| < \sqrt{3} \cdot |k_1 - k_2| \ .$$

(ii) Furthermore, $\mathsf{P}_g > 0$ on $\mathring{H}$ if $k_1 + k_2 \geqslant 0$.

Proof.

(i) According to Example 1.14 (ii),

$$Q_g = -k_1^2 - k_2^2 + \frac{2}{3}(k_1 + k_2)^2 = -\frac{1}{2}(k_1 - k_2)^2 + \frac{1}{6}(k_1 + k_2)^2 \ .$$

(ii) For $i = 1, 2$, denote by $\mathsf{P}_i = -\Delta_i$ the negative of the Laplacian on the manifold M_i. Then, by Proposition 1.5 (ii),

$$\begin{aligned} \mathsf{P}_g &= (\mathsf{P}_1 + \mathsf{P}_2)^2 - 2k_1\mathsf{P}_1 - 2k_2\mathsf{P}_2 + \frac{4}{3}(k_1 + k_2)(\mathsf{P}_1 + \mathsf{P}_2) \\ &= \mathsf{P}_1(\mathsf{P}_1 - 2k_1) + \mathsf{P}_2(\mathsf{P}_2 - 2k_2) + 2\mathsf{P}_1\mathsf{P}_2 + \frac{4}{3}(k_1 + k_2)(\mathsf{P}_1 + \mathsf{P}_2) \geqslant 0 \end{aligned}$$

according to the Lichnerowicz estimate $\mathsf{P}_i \geqslant 2k_i$ for $i = 1, 2$ (which is valid independent of the sign of k_i). Indeed, P_g is positive since the term $2\mathsf{P}_1\mathsf{P}_2$ is positive on the grounded L^2-space. □

7 | OUTLOOK: DISCRETE MODELS AND SCALING LIMITS

In [23], we study discrete approximations of the co-polyharmonic Gaussian field h on the continuous torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and of the associated LQG measure $\mu = \mu_g^{\gamma h}$ on $\mathbb{T}^n$ for arbitrary even n and $|\gamma| < \sqrt{2n}$. The co-polyharmonic Gaussian field h_L on the discrete torus $\mathbb{T}_L^n = \frac{1}{L}\mathbb{Z}^n/\mathbb{Z}^n$ for

$L \in \mathbb{N}$ will be defined in complete analogy to the continuous case. Indeed, however, it also admits an instructive characterization as random field on $\mathbb{R}^{\mathbb{T}_L^n}$ with law explicitly given by

$$c_n\, e^{-b_n \left\| (-\Delta_L)^{n/4} h \right\|^2} dh,$$

where dh is the Lebesgue measure and Δ_L is the discrete Laplacian. The associated discrete LQG measure is the random measure on $\mathbb{T}_L^n$ given as

$$\mu_L(dz) = \exp\left(\gamma h_L(z) - \frac{\gamma^2}{2}\mathbf{E} h_L(z)\right) dz.$$

As $L \to \infty$, we prove convergence of the fields h_L to the co-polyharmonic Gaussian field h on the continuous torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, as well as convergence of the random measures μ_L to the LQG measure μ on $\mathbb{T}^n$ for all $|\gamma| < \sqrt{2n}$.

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