Central convergence on Wiener chaoses always implies asymptotic smoothness and $\mathcal{C}^\infty$ convergence of densities

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Abstract

Let $(F_n)_{n\geq 1}$ be any sequence of Wiener chaoses of fixed order converging in distribution towards a standard Gaussian. In this article, without any additional assumptions, we derive the asymptotic smoothness of the densities of F_n, as well as the convergence of all its derivatives in every $L^q(\mathbb{R})$ for all $q \in [1, +\infty]$ towards the corresponding derivatives of the Gaussian density. In particular, our findings greatly improve the currently known types of convergence which are total variation and entropy.

Finding conditions ensuring that $(F_n)_{n\geq 1}$ converges in distribution to a Gaussian limit and studying metrics and speeds of convergence is an important and broad topic in the recent literature. As an illustration, it is well-known that such convergence is established in total variation [13] and entropy [16], subject to the remarkably simple condition that $\mathbb{E}\left[F_n^4\right] \to 3$ and $\mathbb{E}\left[F_n^2\right] \to 1$. Nevertheless, until now, any attempts to go beyond these types of convergence have imposed significant additional assumptions, such as the requirement of finite negative moments for the Malliavin derivatives, which is a prototypical issue in Malliavin calculus. We refer to [6, 12, 7] or [9, proposition 5.5] for related discussions. In this article, we obtain unconditionally the required control of the negative moments of Malliavin derivatives as we approach Gaussianity in distribution.

Our proof relies on a novel decoupling procedure based on the iteration of an adequate Malliavin gradient enabling to manipulate random quadratic forms in Gaussian variables. Then, the non-degeneracy of the Malliavin derivative is translated in terms of some spectral quantities on a random matrix whose entries are Wiener chaoses of lower order and which measure the proximity with low rank matrices.

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Introduction

In a groundbreaking contribution, Nualart and Peccati [19] made the following surprising discovery, known now as the fourth moment Theorem. For any sequence of random variables $(F_n)_{n\geq 1}$ living in a Gaussian chaos of fixed order,

$$F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1) \Leftrightarrow \mathbb{E}\left[F_n^4\right] \to 3 \,\text{and}\, \mathbb{E}\left[F_n^2\right] \to 1.$$

Later on, in [13], relying on the Stein's method, the metric of convergence in this theorem has been upgraded to total variation. Denoting by $(f_n)_{n\geq 1}$ the densities of $(F_n)_{n\geq 1}$, it is proved

$$F_n \xrightarrow[n\to\infty]{\text{TV}} \mathcal{N}(0,1) \Leftrightarrow \int_{\mathbb{R}} |f_n(x) - \gamma(x)|dx \to 0 \Leftrightarrow F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1).$$

In this article, we make a significant improvement of the latter by establishing the asymptotic smoothness of $(f_n)_{n\geq 1}$ as well as the convergence of all its derivatives towards the corresponding derivatives of the Gaussian density γ in every $L^q(\mathbb{R})$ for $q \in [1,+\infty]$.

$$\begin{aligned} F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1) \quad &\Leftrightarrow \quad \forall q \geq 0, \forall p \in [1,+\infty], f_n^{(q)} \xrightarrow[n\to\infty]{L^p(\mathbb{R})} \gamma^{(q)}. \\ &\Leftrightarrow \quad \forall q \geq 0, \forall p \in [1,+\infty], f_n \xrightarrow[n\to\infty]{W^{q,p}} \gamma. \end{aligned}$$

Note that, in the above right hand side, $f_n^{(q)}$ is only defined for sufficiently large values of n. We will actually establish an equivalent reformulation using estimates of the characteristic functions, see the remark 3 below. We emphasize that in general the densities of random variables in Wiener chaoses are in general absolutely not smooth, even in the simplest case of the second Wiener chaos, hence the above phenomenon is intimately related with central convergence.

Let us describe below the history surrounding the discovery of the fourth moment Theorem.

A brief history

The fourth moment Theorem, has sparked a highly active and fruitful line of research. It has notably led to a significant simplification of the method of moments, which previously involved obtaining the convergence of all moments and is a widespread alternative for establishing a central limit theorem for non-linear functions of a Gaussian field. Among the most notable developments of this breakthrough, regarding Wiener chaoses only, are the following non exhaustive contributions.

[20]: In this article, given a sequence of random vectors $(\vec{F_n})_{n\geq 1} = (F_{n,1}, \cdots, F_{n,d})_{n\geq 1}$ whose coordinates $F_{n,i}$ are in Wiener chaoses of different orders, Peccati and Tudor proved that if for every $i \in \{1,\cdots,d\}$, $F_{n,i} \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,\sigma_i^2)$, then $\vec{F_n}$ converges in distribution towards a Gaussian vector with a diagonal covariance matrix $(\sigma_1^2,\cdots,\sigma_d^2)$. It has resulted in a paradigmatic approach of central convergence for non linear functions of a Gaussian field by computing the orthogonal expansion in Wiener chaoses and establishing the central convergence of each projection, for instance

through the simple criterion provided by [19]. Then, one is left to use the fact that separate convergence entails joint convergence and thus the convergence of the sum of the projections, hence the convergence of the sequence itself.

[18]: In this article, Latorre and Nualart established that $F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1)$ if and only if the Malliavin derivative $\langle DF_n, DF_n \rangle_{\mathcal{H}}$, or equivalently the square field operator $\Gamma[F_n, F_n]$, tends in L^2 towards a constant.

[13]: In this article, in connection with the so-called Stein's method, Nourdin and Peccati established the following quantitative version for every $F \in \mathcal{W}_m$ with $\mathbb{E}[F^2] = 1$:

$$\mathrm{d_{TV}}(F, \mathcal{N}(0,1)) \leq \frac{2}{p}\sqrt{\operatorname{Var}[\Gamma[F,F]]} \tag{1}$$

$$\leq \frac{2}{\sqrt{3}}\sqrt{\mathbb{E}[F^4 - 3]}. \tag{2}$$

This landmark contribution has catalysed a fertile line of research that centres on the symbiotic interplay between Stein's method, which characterizes and quantifies convergence in distribution via certain differential operators, and Malliavin calculus, which furnishes a panoply of differential operators and a framework for integration by parts. One may consult the constantly updated webpage `https://sites.google.com/site/malliavinstein/home` for a comprehensive overview of this area of research.

[8, 1]: In these two recent contributions a new proof of the fourth moment theorem is provided, using a novel approach that leverages the rich spectral properties of Wiener chaoses. These properties arise from the fact that Wiener chaoses are also eigenfunctions of the Ornstein-Uhlenbeck generator. This spectral perspective avoids the need of product formulas for Wiener chaoses, which are also known as multiple Wiener-Itô integrals, and sheds new light on moment inequalities for Wiener chaoses in relation to the Gaussian product conjecture. For an overview of these developments, see [10] and [2].

[16]: In this contribution, using the so-called De Bruijn's formula, Nourdin Peccati and Swan established that the fourth moment Theorem holds in entropy. Via the Pinsker's inequality this is strictly better than the convergence in total variation as established in [13], though the rates of convergence are slightly weaker due to the presence of an additional logarithmic term. This was the strongest sense of convergence for *general sequences of Wiener chaoses* that is available in the literature.

Going beyond entropy: getting negative moments for $\Gamma[F_n, F_n]$

Hitherto, any attempt to provide better metrics for the convergence of a sequence of Wiener chaoses towards a Gaussian limit, that is to say going beyond entropic convergence, has been confronted with the imperious necessity of bouding negative moments for the quantity $\Gamma[F_n, F_n] = \langle DF_n, DF_n \rangle_{\mathcal{H}}$.

- For instance in [6], assuming that $F_n \to \mathcal{N}(0,1)$, it is proved that under the assumption that

$$\forall p \geq 1, \limsup_{n\to\infty} \mathbb{E}\left[\frac{1}{\Gamma\left[F_n, F_n\right]^p}\right] < \infty,$$

then the sequence of densities $(f_n)_{n\geq 1}$ corresponding to the sequence of Wiener chaoses $(F_n)_{n\geq 1}$ converges towards $\gamma(x) := e^{-\frac{x^2}{2}}/\sqrt{2\pi}$ in $\mathcal{C}^\infty$ topology. More precisely, it is proved that

$$\forall p \geq 0, \sup_{x\in\mathbb{R}} \left| f_n^{(p)}(x) - \gamma^{(p)}(x) \right| \to 0.$$

- In the same vein, in [12], it is proved that the fourth moment Theorem holds in the sense of the Fisher information whenever

$$\limsup_{n\to\infty} \mathbb{E}\left[\Gamma\left[F_n, F_n\right]^{-4-\epsilon}\right] < \infty,$$

  and some examples are provided in the case of Wiener chaoses of order two which (via diagonalization arguments) can always be written as a weighted sum of independent centred chi-squares. Hence, depending on the behaviour of these weights it is indeed possible to check whether one has negative moments for the Malliavin derivatives of the sequence. We also refer to [9, proposition 5.5] for related bounds involving the Fisher information together with the assumption of negative moments for the square field operator.

- One may also cite the article [7] where the $\mathcal{C}^\infty$ convergence of densities is established for properly rescaled Hermite sums of a stationary Gaussian field under mild assumptions on the spectral densities. Using the very specific shape of this kind of Wiener chaoses, the authors were able to prove negative moments for the Malliavin derivatives and hence the result.

Main results

In this article, exploiting finely the rich structure of Wiener chaoses, we establish that the recurrent assumption of negative moments for the quantity $\langle DF_n, DF_n\rangle_{\mathcal{H}}$ is entirely unnecessary as it follows solely from the central convergence of the sequence $(F_n)_{n\geq 1}$. This remarkable phenomenon of regularization has gone unnoticed until now and allows for a significant enhancement of the central convergence on Wiener chaoses, from total variation or entropy to $\mathcal{C}^\infty$ convergence of densities, as precisely stated in Theorem 2 and remark 3. While it is already known from [18, 13] that central convergence guarantees the convergence towards a constant of the Malliavin derivatives in L^p spaces for $p \geq 1$, we provide additional information for negative integers p, enabling full deployment of Malliavin calculus and the iteration of integrations by parts for regularization.

We first express our first result in the language of Malliavin calculus through the behaviour of the Malliavin derivative/square field operator $\Gamma[F_n, F_n]$ which gets negative moments of arbitrarily high orders as we get closer to Gaussian in distribution.

Theorem 1. *Let us fix $m \geq 1$ and let us consider $(F_n)_{n\geq 0}$ a sequence of random variables in the m-th Wiener chaos $\mathcal{W}_m$. We assume that $F_n \xrightarrow[n\to\infty]{Law} \mathcal{N}(0,1)$. Then for every $q \geq 1$*

there exists an integer N_q and a positive constant C_q such that

$$\sup_{n \geq N_q} \mathbb{E}\left[\frac{1}{\Gamma\left[F_n, F_n\right]^q}\right] \leq C_q. \tag{3}$$

Remark 1. *It is possible to reformulate the previous result in a non sequential way. Let us fix here m the order of the considered chaos. In order to state our results we need a distance supporting the convergence in distribution. We choose here the so-called Fortet-Mourier distance, defined by*

$$d_{FM}(U, V) := \sup_{\|\phi\|_\infty \leq 1, \|\phi'\|_\infty \leq 1} \left|\mathbb{E}\left[\phi(U) - \phi(V)\right]\right|. \tag{4}$$

This specific choice plays no role in the sequel. For every $q \geq 1$ there exists $\delta_q > 0$ and $C_q > 0$ such that for every F in $\mathcal{W}_m$:

$$d_{FM}(F, \mathcal{N}(0,1)) \leq \delta_q \Rightarrow \mathbb{E}\left[\frac{1}{\Gamma[F, F]^q}\right] \leq C_q. \tag{5}$$

Following the approaches in [6, 4] or else [7, proposition 2.6], our theorem implies the next theorem.

Theorem 2. *Let $(F_n)_{n \geq 1}$ be a sequence in $\mathcal{W}_m$ converging in law to the standard Gaussian. Then, denoting f_n the density of F_n (which always exists, see for instance [17]) and γ be the density of the standard Gaussian distribution, for every integer $q \geq 0$, we have that for n large enough, $f_n^{(q)}$ is well defined, and $f_n^{(q)} \to \gamma^{(q)}$ in $L^1(\mathbb{R})$. This is actually equivalent to $f_n \xrightarrow[n \to \infty]{W^{p,q}} \gamma$ for every $q \geq 0$ and $p \in [1, +\infty]$.*

Remark 2. *It is standard that the above conclusion for $q = 1$ implies the convergence $f_n \to \gamma$ in $L^p(\mathbb{R})$ for every $p \leq +\infty$, which was already unknown at our knowledge. Actually in the same way we even have for n large, $f_n \in \mathcal{C}^q$ and $f_n^{(q)} \to \gamma^{(q)}$ in $L^p(\mathbb{R})$ for any $p \leq +\infty$.*

Our theorem also gives estimates on characteristic functions of variables in Wiener chaos close to a Gaussian variable in distribution, which may be advantageous in some circumstances.

Remark 3. *For every $q \geq 1$ there exists $\delta_q > 0$ and $C_q > 0$ such that for every $F \in \mathcal{W}_m$ we have*

$$d_{FM}(F, \mathcal{N}(0,1)) \leq \delta_q \Rightarrow \sup_{|t| \geq 1} |t|^q \left|\mathbb{E}\left[e^{itF}\right]\right| \leq C_q. \tag{6}$$

It is important to note that our results are qualitative in nature and determining the correct dependence of the constants in our proofs would be a demanding but worthwhile endeavour.

A few consequences

We describe below some potential applications of our findings that we aim at developing and exploring in forthcoming contributions.

- The celebrated Carbery-Wright inequality, see [3], implies for $(G_1, \ldots, G_n)$ a standard Gaussian vector and P is a degree-d polynomial with $\mathbb{E}[|P(G_1, \ldots, G_n)|] = 1$, that
$$\mathbb{P}\left(|P(G_1, \cdots, G_n)| \leq \epsilon\right) \leq c_d \epsilon^{\frac{1}{d}},$$
where c_d depends only on d, not on n. In the case of multilinear homogeneous sums, the above inequality has been used in the seminal contribution [11] to obtain quantitative invariance principles for various convergence metrics, with resulting bounds that may depend on d for the roughest metrics. A multilinear homogeneous sum evaluated in a standard Gaussian vector is an archetypal example of Wiener chaos, so our Theorem 2 may apply. Thus, if $\mathrm{d}_{FM}(P(G_1, \cdots, G_n), \mathcal{N}(0,1))$ is small enough, then $P(G_1, \cdots, G_n)$ has a bounded density and
$$\mathbb{P}(|P(G_1, \cdots, G_n)| \leq \epsilon) \leq c_d \epsilon,$$
for an another constant c_d. This considerably improves the exponent on ϵ.

- Theorem 1 can be associated with [6, Theorem 4.4] in order to establish that for every $p \geq 0$ there exists positive constants C_p, α_p (depending on p) such that for any Wiener chaos F of order m:
$$\mathbb{E}\left[F^4 - 3\right] < \alpha_p \Rightarrow f_F \in \mathcal{C}^p \text{ and } \sup_{x\in\mathbb{R}} \left| f_F^{(p)}(x) - \gamma^{(p)}(x) \right| \leq C_p \sqrt{\mathbb{E}\left[F^4 - 3\right]}. \quad (7)$$
Again, it would be very interesting and useful to give more quantitative statements for the constants α_p, C_p but this task would be rather demanding and falls beyond the scope of this article.

- Theorem 1 can also be associated with [12, theorem 1.1] in order to establish at the neighbourhood of Gaussianity that
$$J(F) - 1 \leq c_p \left(\mathbb{E}\left[F^4\right] - 3\right)$$
where $J(F)$ denotes the so-called Fisher information. Relying [12, equation 1], the latter bounds the relative entropy. This in particular implies that
$$\exists C_p > 0,\, \forall F \in \mathcal{W}_m \text{ with } \mathbb{E}\left[F^2\right] = 1,\; D(F||N) \leq C_p \sqrt{\mathbb{E}\left[F^4 - 3\right]}.$$
This improves the bound given in [16] which contains a superfluous logarithmic term, at least in dimension one.

- At first glance, it is not immediate that our findings apply to random vectors with chaotic components. However, in the particular case where the components lie in a Wiener chaos of the same order, we may establish as well the $\mathcal{C}^\infty$ convergence of the densities. Indeed, noting $\vec{F} = (F_1, \cdots, F_d)$ where $F_i \in \mathcal{W}_m$ for every $i \in \{1, \cdots, d\}$, the equation (6) applied to $\langle \frac{\vec{\xi}}{\|\vec{\xi}\|}, \vec{F} \rangle$ which belongs to $\mathcal{W}_m$ immediately gives that
$$d_{\mathrm{FM}}(\vec{F}, \mathcal{N}(0, \mathrm{I}_d)) \leq \delta_q \Rightarrow \sup_{|\vec{\xi}| \geq 1} \|\vec{\xi}\|_2^q \left| \mathbb{E}\left[e^{i\langle \vec{\xi}, \vec{F} \rangle}\right] \right| \leq C_q.$$
It is certainly of interest to investigate the whole multivariate counterpart of our findings, however this is beyond the scope of this article and will be treated separately.

1 Preliminaries

In this section, we provide the necessary definitions/notations and preliminary lemmas that are required for the proof of the main theorem. In the whole article for any parameter α, C_α stands for a constant which only depends on α and may possibly change from line to line. Nevertheless, for the sake of clarity, we will generally not track the dependence on m denoting the order of the chaoses.

1.1 Prolegomena on Wiener chaos and Malliavin calculus

1.1.1 Wiener chaoses in a nutshell

Let us describe below the strict necessary knowledge on Wiener chaoses that we will use throughout the article. We refer to [14] for a more detailed exposition.

The Wiener space: We place ourselves on the following countable product of probability spaces which is refer as to the Wiener space.

$$(\Omega, \mathcal{F}, \mathbb{P}) := (\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)^{\mathbb{N}}.$$

Let us define for all $i \geq 0$:

$$N_i := \begin{cases} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)^{\mathbb{N}} & \longrightarrow & \mathbb{R} \\ (x_0, x_1, \cdots) & \longrightarrow & x_i \end{cases}$$

By construction, $(N_i)_{i\geq 0}$ are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ which are i.i.d. and follow a standard Gaussian distribution. We shall sometimes require auxiliary sequences of independent standard Gaussian random variables $(G_i, H_i)_{i\geq 1}$ themselves independent of $(N_i)_{i\geq 1}$. They may be built in the same way as coordinates of auxiliary Wiener spaces and one is left to work for instance on the augmented Wiener space

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)^{\mathbb{N}} \times (\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)^{\mathbb{N}} \times (\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)^{\mathbb{N}}.$$

Since $\mathbb{N}^3$ is equipotent with $\mathbb{N}$, the augmented Wiener space is actually a Wiener space.

Wiener chaoses: The m-th Wiener chaos may be defined in the following way

$$\mathcal{W}_m := \mathrm{Adh}_{L^2(\mathbb{P})} \left(\mathrm{Vect} \left(\prod_{i=0}^{\infty} H_{k_i}(x_i) \mid \sum_{i=0}^{\infty} k_i = m \right) \right). \tag{8}$$

There $(H_k(x))_{k\geq 0}$ denotes the k-th Hermite polynomial and the above infinite product is actually finite as $H_0(x) = 1$ and only finitely many integers $(k_i)_{i\geq 0}$ are non zero. One important feature of Wiener chaoses is that

$$L^2(\mathbb{P}) = \bigoplus_{m=0}^{\infty} \mathcal{W}_m.$$

In order to avoid the use of infinite dimensional operators and manipulate instead finite matrices it will be convenient for us to work with *finitely generated* Wiener chaoses defined as

$$\mathcal{W}_m^{(0)} := \mathrm{Vect}\left(\prod_{i=0}^{\infty} H_{k_i}(x_i) \mid \sum_{i=0}^{\infty} k_i = m\right). \tag{9}$$

In the next subsection we make precise our requirements on our sequence of random variables $(F_n)_{n\geq 1}$.

1.1.2 Working with finitely generated Wiener chaoses

Though our theorem and our corollaries are stated for general Wiener chaoses, one may work on $\mathcal{W}_m^{(0)}$ instead of $\mathcal{W}_m$. Indeed, let $q \geq 1$, by establishing Theorem 1 for $\mathcal{W}_m^{(0)}$ one gets $\delta_q, C_q > 0$ such that

$$\forall F \in \mathcal{W}_m^{(0)}, d_{\mathrm{FM}}(F, \mathcal{N}(0,1)) < \delta_q \Rightarrow \mathbb{E}\left[\frac{1}{\Gamma[F,F]^q}\right] \leq C_q.$$

However, $\mathcal{W}_m^{(0)}$ is dense on $\mathcal{W}_m$, hence the result. This justifies the following definition.

Definition 1. *Throughout the whole article, $(F_n)_{n\geq 1}$ will denote a sequence of elements in $\mathcal{W}_m^0$ such that $\mathbb{E}\left[F_n^2\right] = 1$ and $(k_n)_{n\geq 1}$ will be a sequence of integers such that*

$$\forall n \geq 1, F_n \in \mathbb{R}_m\left[N_1, \cdots, N_{k_n}\right].$$

1.1.3 Malliavin calculus operators

Here we simply introduce the Malliavin operators that we use in this article. We refer for instance to [14] for a much more complete introduction. We emphasize that we shall only deal with polynomial mappings evaluated in finitely dimensional Gaussian vectors so that we may entirely skip any domain/integrability considerations that are frequent in Malliavin calculus.

<u>The square field operator:</u>

Throughout the sequel, given a standard Gaussian vector $\vec{N} = (N_1, \cdots, N_d)$ and a polynomial mapping $f \in \mathbb{R}\left[N_1, N_2, \cdots, N_d\right]$ we define the so-called *square field operator* or $F = f(N_1, \cdots, N_d)$ by

$$\Gamma\left[F, F\right] = \sum_{i=1}^{d} \left(\frac{\partial f}{\partial N_i}(\vec{N})\right)^2. \tag{10}$$

When considering for instance $F = f(N_1, \cdots, N_d, G_1, \cdots, G_d)$ where $\vec{N}$ and $\vec{G}$ are independent standard Gaussian vectors, we set

$$\begin{aligned}
\Gamma_N\left[F, F\right] &= \sum_{i=1}^{d} \left(\frac{\partial f}{\partial N_i}(\vec{N}, \vec{G})\right)^2 \\
\Gamma_G\left[F, F\right] &= \sum_{i=1}^{d} \left(\frac{\partial f}{\partial G_i}(\vec{N}, \vec{G})\right)^2,
\end{aligned}$$

so that $\Gamma[F,F] = \Gamma_N[F,F] + \Gamma_G[F,F]$.

The sharp operator:

Let us consider a standard Gaussian vector $\vec{N} = (N_1, \cdots, N_d)$ and a polynomial mapping $f \in \mathbb{R}[N_1, N_2, \cdots, N_d]$ as well as $\vec{G} = (G_1, \cdots, G_d)$ an independent standard Gaussian vector. We then set

$$\sharp_G[F] = \sum_{i=1}^{d} \frac{\partial f}{\partial N_i}(\vec{N}) G_i. \tag{11}$$

This operator is intimately related with the square field operator through the Laplace-Fourier identity

$$\mathbb{E}\left[e^{it\sharp_G[F]}\right] = \mathbb{E}\left[\exp\left(-\frac{t^2}{2}\Gamma[F,F]\right)\right].$$

Link between central convergence and square field operator for Wiener chaoses:

At several moments of the article we shall use the following facts which are well-known in the literature.

- For any sequence of Wiener chaoses of fixed order m we have the equivalence

  $$F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1) \Leftrightarrow \Gamma[F_n, F_n] \xrightarrow[n\to\infty]{L^2(\mathbb{P})} m.$$

  We refer for instance to [13] for a proof.

- We shall also rely on the multivariate extension of the latter. Let us consider $\vec{F_n} = (F_{n,1}, \cdots, F_{n,d})$ a sequence of random vectors whose coordinates are Wiener chaoses of fixed orders $(m_1, \cdots, m_d)$ such that $\vec{F_n} \to \mathcal{N}(0, \mathrm{I}_d)$. Then, it is equivalent to saying that

  $$\forall i \neq j,\ \Gamma[F_{n,i}, F_{n,j}] \xrightarrow[n\to\infty]{L^2(\mathbb{P})} 0 \text{ and } \Gamma[F_{n,i}, F_{n,i}] \xrightarrow[n\to\infty]{L^2(\mathbb{P})} m_i.$$

  One may consult [15, proposition 3.10] for a proof.

1.1.4 Polynomially fast convergence of characteristic functions

Instead of working with infinite norms on the densities, it will be more convenient for us to express our conditions through the decay of characteristic functions.

Definition 2. *For any random variable Y and any sequence of random variable $(X_n)_{n\geq 0}$ we say that $X_n \xrightarrow[n\to\infty]{\mathcal{F}^q} Y$ if and only if*

$$\sup_{|t|\geq 1} \left| t^q \mathbb{E}\left[e^{itX_n} - e^{itY}\right]\right| \xrightarrow[n\to\infty]{} 0. \tag{12}$$

When Y admits a density which is of class $W^{q,\infty}$, the previous mode of convergence guarantees the convergence of the densities of X_n towards the density of Y in the Sobolev space $W^{q,\infty}$.

We shall require the following lemma whose proof proceeds from the iteration of the integration by parts formula. This lemma is not new and is rather standard in Malliavin calculus. One may consult for instance the reference [6, Theorem 4.4]. This result can be stated in a much more general form, but for simplicity we restrict ourselves to the case of a sequence of chaotic random variables converging towards a Gaussian limit, which is enough for our purpose. We refer to [6] or [4] for a proof.

Lemma 1. *Let $(F_n)_{n\geq 1}$ be a sequence of random variables in the m-th Wiener chaos which converges towards a standard Gaussian limit. Let us assume that for every $q \geq 1$ we have $\limsup_n \mathbb{E}\left[\Gamma\left[F_n, F_n\right]^{-q}\right] < \infty$. Then, one gets*

$$\forall q \geq 1,\ F_n \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0,1).$$

1.1.5 Decoupling twice on Wiener chaoses

It will be convenient for us to work with the following decoupled version of F_n, which up to a multiplicative constant corresponds to $\sharp_H\left[\sharp_G\left[F_n\right]\right]$, where this operator is defined in equation (11).

Definition 3. *Let $(G_n, H_n)_{n\geq 1}$ an independent sequence of standard Gaussian couples. We set*

$$\begin{aligned} \widetilde{F_n} &= \frac{1}{\sqrt{m(m-1)}} \sum_{i,j\geq 1} \frac{\partial^2 F_n}{\partial N_i \partial N_j} G_i H_j \\ &= \sum_{i,j\geq 1} A_n(i,j) G_i H_j. \end{aligned}$$

Here, A_n is a symmetric matrix of dimension k_n whose entries are Wiener chaoses of order $m-2$. $\widetilde{F_n}$ may be seen as a quadratic form evaluated in the random vector $(G_1, \cdots, G_n, H_1, \cdots, H_n)$ which is associated with the matrix $\widetilde{A_n}$ given by

$$\widetilde{A_n} = \begin{pmatrix} 0 & A_n \\ A_n & 0 \end{pmatrix},$$

so that

$$\widetilde{F_n} = (G_1, \cdots, G_n, H_1, \cdots, H_n)\widetilde{A_n}{}^{\intercal}(G_1, \cdots, G_n, H_1, \cdots, H_n).$$

Remark 4. *One may prove that the characteristic polynomial of $\widetilde{A_n}$ is $\chi_{A_n}(t)\chi_{A_n}(-t)$ where χ_{A_n} stands for the characteristic polynomial of A_n. Hence, the spectrum of $\widetilde{A_n}$ is simply the duplication (up to the sign) of the spectrum of A_n.*

Remark 5. *Since $\mathbb{E}\left[F_n^2\right] = 1$, one may check that $\mathbb{E}\left[\widetilde{F_n}^2\right] = 1$. Indeed, for any $F \in \mathcal{W}_m$ the integration by parts leads to $m\mathbb{E}\left[F^2\right] = \mathbb{E}\left[\Gamma\left[F, F\right]\right]$. Hence we have*

$$\begin{aligned} 1 &= \mathbb{E}\left[F_n^2\right] = \frac{1}{m}\mathbb{E}\left[\Gamma\left[F_n, F_n\right]\right] \\ &= \frac{1}{m}\sum_{k=1}^{k_n} \mathbb{E}\left[\left(\frac{\partial F_n}{\partial N_k}\right)^2\right] \quad \left(\textit{with } \frac{\partial F_n}{\partial N_k} \in \mathcal{W}_{m-1}\right) \\ &= \frac{1}{m(m-1)} \sum_{i,j=1}^{k_n} \mathbb{E}\left[\left(\frac{\partial^2 F_n}{\partial N_i \partial N_j}\right)^2\right] = \mathbb{E}\left[\widetilde{F_n}^2\right]. \end{aligned}$$

The following proposition ensures the remarkable equivalence between the central convergence of $(F_n)_{n\geq 1}$ and the central convergence of $(\widetilde{F_n})_{n\geq 1}$ and is of independent interest.

Proposition 1.

$$\widetilde{F_n} \xrightarrow[n\to\infty]{Law} \mathcal{N}(0,1) \Leftrightarrow F_n \xrightarrow[n\to\infty]{Law} \mathcal{N}(0,1). \tag{13}$$

Proof. We start by the following relation relating the square field operator of F_n with $\sharp_G[F_n]$.

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\frac{t^2}{2}\Gamma\left[F_n,F_n\right]\right)\right] &= \mathbb{E}\left[\exp\left(-\frac{t^2}{2}\sum_{k=1}^{k_n}\left(\frac{\partial F_n}{\partial N_k}\right)^2\right)\right] \\
= \mathbb{E}\left[\exp\left(it\sum_{k=1}^{k_n}\frac{\partial F_n}{\partial N_k}G_k\right)\right] &= \mathbb{E}\left[\exp\left(it\sharp_G\left[F_n\right]\right)\right],
\end{aligned}$$

where we recall that $(G_k)_{k\geq 1}$ is independent of $(N_k)_{k\geq 1}$. For any functional smooth in the Malliavin sense of the form

$$Z = f(N_1,\cdots,N_{k_n},G_1,\cdots,G_{k_n})$$

we recall that

$$\Gamma_N\left[Z,Z\right] = \sum_{k=1}^{k_n}\left(\frac{\partial Z}{\partial N_k}\right)^2, \ \Gamma_G\left[Z,Z\right] = \sum_{k=1}^{k_n}\left(\frac{\partial Z}{\partial G_k}\right)^2.$$

Hence, we derive the following equivalences which exploit that on Wiener chaoses the central convergence is equivalent to the fact that the square field operator tends to a constant (see [18, 13]).

$$\begin{aligned}
F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1) &\Leftrightarrow \Gamma\left[F_n,F_n\right] \xrightarrow[n\to\infty]{L^2} m, \\
&\Leftrightarrow \sharp_G\left[F_n\right] \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,m), \\
&\Leftrightarrow \Gamma_N\left[\sharp_G\left[F_n\right],\sharp_G\left[F_n\right]\right] + \underbrace{\Gamma_G\left[\sharp_G\left[F_n\right],\sharp_G\left[F_n\right]\right]}_{=\Gamma[F_n,F_n]} \to m^2, \\
&\Leftrightarrow \Gamma_N\left[\sharp_G\left[F_n\right],\sharp_G\left[F_n\right]\right] \to m^2 - m.
\end{aligned}$$

However, using a Laplace-Fourier identity we also get that

$$\mathbb{E}\left[\exp\left(-\frac{t^2}{2}\Gamma_N\left[\sharp_G\left[F_n\right],\sharp_G\left[F_n\right]\right]\right)\right] = \mathbb{E}\left[\exp\left(it\sum_{i,j=1}^{k_n}\frac{\partial^2 F_n}{\partial N_i\partial N_j}G_iH_j\right)\right].$$

Combining these results leads to the final equivalence:

$$\begin{aligned}
F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1) &\Leftrightarrow \sum_{i,j=1}^{k_n}\frac{\partial^2 F_n}{\partial N_i\partial N_j}G_iH_j \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,m(m-1)) \\
&\Leftrightarrow \widetilde{F_n} \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1).
\end{aligned}$$

□

Proposition 2. *If we assume that for any $q \geq 1$ we have $\widetilde{F_n} \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0,1)$ then the conclusion of the Theorem 1 holds true. That is to say, for every $q \geq 1$ there exists $N_q \in \mathbb{N}$ and $C_q > 0$ such that*

$$\sup_{n\geq N_q} \mathbb{E}\left[\frac{1}{\Gamma[F_n, F_n]^q}\right] \leq C_q.$$

Proof. We will divide the proof in several distinct steps.

Step 1: We prove in this step that:

$$\forall q \geq 1, \limsup_{n\to\infty} \mathbb{E}\left[\frac{1}{\Gamma_N\left[\sharp_G\left[F_n\right], \sharp_G\left[F_n\right]\right]^q}\right] < \infty.$$

Let us fix $q \geq 1$. Since we assume that $\widetilde{F_n} \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0,1)$, by definition of this mode of convergence there exists N_q large enough and a constant C_q which may vary from line to line such that

$$\forall n \geq N_q, \forall t \in \mathbb{R}, \left|t^{2q-1}\mathbb{E}\left[e^{it\widetilde{F_n}}\right]\right| < \frac{C_q}{t^2+1}.$$

Due to Fourier-Laplace identities, this in particular implies that

$$\forall n \geq N_q, \forall t \in \mathbb{R}, \left|t^{2q-1}\mathbb{E}\left[\exp\left(-\frac{t^2}{2}\Gamma_N\left[\sharp_G\left[F_n\right], \sharp_G\left[F_n\right]\right]\right)\right]\right| < \frac{C_q}{t^2+1}.$$

Then, it is enough to remark that for any strictly positive real number A we have

$$\begin{aligned}\int_{\mathbb{R}} |t^{2q-1}| \exp\left(-\frac{t^2}{2}A\right) dt &\overset{s=\sqrt{A}t}{=} \int_{\mathbb{R}} \frac{|s^{2q-1}|}{A^{\frac{2q-1}{2}}} \exp\left(-\frac{s^2}{2}\right) \frac{ds}{\sqrt{A}} \\ &= \frac{C_q}{A^q}.\end{aligned}$$

Applying the previous estimate with $A = \Gamma_N\left[\sharp_G\left[F_n\right], \sharp_G\left[F_n\right]\right]$ and taking the expectation leads to the desired claim.

Step 2: Here we prove that for every $q \geq 1$ we have $\sharp_G[F_n] \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0,m)$.

First of all, by the proof of the previous proposition we already know that $\sharp_G[F_n]$ tends in distribution to $\mathcal{N}(0,m)$. Interpreting $\sharp_G[F_n]$ as an element of the m-th Wiener chaos generated by the Gaussian sequences $(N_n, G_n)_{n\geq 1}$, we have

$$\Gamma\left[\sharp_G[F_n], \sharp_G[F_n]\right] = \Gamma_N\left[\sharp_G[F_n], \sharp_G[F_n]\right] + \Gamma_G\left[\sharp_G[F_n], \sharp_G[F_n]\right] \geq \Gamma_N\left[\sharp_G[F_n], \sharp_G[F_n]\right].$$

Hence we get that for every $q \geq 1$ there exists $N_q \in \mathbb{N}$ and a constant C_q such that

$$\sup_{n\geq N_q} \mathbb{E}\left[\frac{1}{\Gamma\left[\sharp_G[F_n], \sharp_G[F_n]\right]^q}\right] \leq C_q.$$

However, $(\sharp_G[F_n])$ may be seen as a sequence of chaotic random variables converging towards a Gaussian of variance 1. Applying the lemma 1, one deduces that

$$\sharp_G[F_n] \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0,m).$$

Step 3: In this final step we establish the desired claim.

One is left to establish negative moments of every order for $\Gamma[F_n, F_n]$ and to apply lemma 1 and conclude. Let $q \geq 1$, we copy the end of the Step 1. By the previous step $\sharp_G[F_n] \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0, m)$ and thus there exists an integer N_q and a constant C_q which possibly vary from line to line such that

$$\forall n \geq N_q, \forall t \in \mathbb{R}, \left|t^{2q-1}\mathbb{E}\left[e^{it\sharp_G[F_n]}\right]\right| < \frac{C_q}{t^2+1}.$$

Then, we write

$$\int_{\mathbb{R}} |t^{2q-1}|\mathbb{E}\left[\exp\left(-\frac{t^2}{2}\Gamma[F_n, F_n]\right)\right] dt \quad = \quad C_q\mathbb{E}\left[\frac{1}{\Gamma[F_n, F_n]^q}\right] \leq \underbrace{C_q \int_{\mathbb{R}} \frac{dt}{t^2+1}.}_{:=C_q},$$

where we recall that C_q may vary from line to line and only depends on q.

□

1.1.6 Conditional central convergence for Wiener chaoses

The next lemma may be seen as a conditional form of the multivariate fourth moment Theorem.

Lemma 2. *Let A_n be the random matrix introduced in definition 3. We give us X_n a sequence of matrices of $\mathcal{M}_{k_n,q}(\mathbb{R})$ whose entries are i.i.d. standard Gaussian random variables, as well as a sequence of random vectors $Y_n \sim \mathcal{N}(0, I_{k_n})$ in such a way that $((X_n)_{n\geq 1}, (Y_n)_{n\geq 1}, (N_i)_{i\geq 1}, (G_i, H_i)_{i\geq 1})$ are jointly independent. Under the assumption that $F_n \xrightarrow[n\to\infty]{Law} \mathcal{N}(0, 1)$, we have*

$$d_{FM}^{Y,A_n}\left({}^\intercal(A_n X_n) Y_n, \mathcal{N}(0, Id_q)\right) \xrightarrow[n\to\infty]{Prob} 0,$$

where the above Forter-Mourier distance is computed through the random variables (Y_n, A_n) only and not X_n, which is suggested by the notation $d_{FM}^{Y_n,A_n}$. It is then a random variable X_n-measurable.

Proof. Let us consider the sequence of q-dimensional random vectors ${}^\intercal(A_n X_n)Y_n$. Since the coefficients of X_n are jointly independent it may be readily checked that the coordinates of ${}^\intercal(A_n X_n)Y_n$ are uncorrelated and separately converge in distribution to a standard Gaussian distribution since they are equal in distribution with $\widetilde{F_n}$.

Let us write ${}^\intercal(A_n X_n)Y_n = (Z_{1,n}, \cdots, Z_{q,n})$. From the multivariate Fourth moment Theorem (see for instance [15, proposition 3.10]) one may deduce that

$$\mathbb{E}_X\left[\mathbb{E}_{A,Y}\left[\sum_{i=1}^{q} |\Gamma[Z_{i,n}, Z_{i,n}] - m|^2 + \sum_{i\neq j} \Gamma\left[Z_{i,n}, Z_{j,n}\right]^2\right]\right] \xrightarrow[n\to\infty]{} 0.$$

There, $\mathbb{E}_X$ denotes the expectation with respect to the random variables X_n and $\mathbb{E}_{A,Y}$ stands for the expectation with respect to A and Y. We recall that these variables are independent jointly so we may separate in this way the expectations. Besides we may write

$$\begin{aligned}\Gamma\left[Z_{i,n},Z_{i,n}\right] &= \Gamma_N\left[Z_{i,n},Z_{i,n}\right]+\Gamma_{A_n,Y}\left[Z_{i,n},Z_{i,n}\right]\\ Z_{i,n} &\overset{\text{Law}}{=} \mathcal{N}\left(0,\Gamma_N\left[Z_{i,n},Z_{i,n}\right]\right)\\ &\xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0,1\right).\end{aligned}$$

Thus, from the previous estimate we may deduce that $\Gamma_N\left[Z_{i,n},Z_{i,n}\right]$ tends to 1 in probability and hence in L^2 by standard arguments of hypercontractivity. Besides for $i\neq j$, we may write

$$\Gamma[Z_{i,n},Z_{j,n}]=\underbrace{\Gamma_N[Z_{i,n},Z_{j,n}]}_{=0}+\Gamma_{A,Y}[Z_{i,n},Z_{j,n}]=\Gamma_{A,Y}[Z_{i,n},Z_{j,n}].$$

Hence, removing the contribution of $\Gamma_N\left[Z_{i,n},Z_{i,n}\right]$ leads to the fact that

$$\mathbb{E}_{A,Y}\left[\sum_{i=1}^{q}\left|\Gamma_{A,Y}[Z_{i,n},Z_{i,n}]-(m-1)\right|^2+\sum_{i\neq j}\Gamma_{A,Y}\left[Z_{i,n},Z_{j,n}\right]^2\right]$$

converges to zero in $L^1(\mathbb{P}_X)$ and thus in probability.

Finally, since $\mathbb{P}_X$-a.s. $Z_{i,n}$ belongs to the $m-1$-th Wiener chaos, it is also well known (see for instance [15, Theorem 3.5]) that the square root of this quantity is an upper bound for the quantity

$$d_{\mathrm{FM}}^{Y_n,A_n}\left({}^{\intercal}\left(A_nX_n\right)Y_n,\mathcal{N}(0,\mathrm{Id}_q)\right)$$

which achieves the proof. □

1.1.7 Spectral considerations

Given a symmetric matrix A with spectrum ordered by decreasing absolute values $|\lambda_1|\geq|\lambda_2|\geq\cdots$, we introduce the following spectral quantities:

- $R_q(A)=\sum_{i_1<i_2<\cdots<i_q}\lambda_{i_1}^2\cdots\lambda_{i_q}^2$,
- $r_q(A)=\sum_{i\geq q}\lambda_i^2$,
- $\mathcal{D}_q(A)=\inf\{\|A-B\|^2\,\big|\,\mathrm{rank}(B)\leq q-1\}$ with $\|C\|=\sqrt{\mathrm{tr}(C^{\intercal}C)}$ being the so-called Frobenius norm.

Lemma 3. *For each integer $q\geq 2$ we have*

$$\frac{1}{q}\,R_{q-1}(A)r_q(A)\leq R_q(A)\leq R_{q-1}(A)r_q(A). \tag{14}$$

$$\frac{1}{q!}\prod_{i=1}^{q}r_i(A)\leq R_q(A)\leq\prod_{i=1}^{q}r_i(A). \tag{15}$$

$$\frac{r_q(A)^q}{q!}\leq R_q(A)\leq r_1(A)^{q-1}r_q(A). \tag{16}$$

Proof. Let us prove first the inequality (14), the second and third inequalities proceed from an immediate induction.

$$R_q(A) = \sum_{i_1<i_2<\cdots<i_{q-1}} \lambda_{i_1}^2 \cdots \lambda_{i_{q-1}}^2 \underbrace{\sum_{i_q>i_{q-1}} \lambda_{i_q}^2}_{\leq r_q(A)} \leq R_{q-1}(A) r_q(A).$$

For the left inequality we write

$$\begin{aligned} R_q(A) &= \frac{1}{q!} \sum_{\substack{(i_1,\cdots,i_q) \\ \text{pairwise distinct}}} \lambda_{i_1}^2 \cdots \lambda_{i_q}^2 \\ &= \sum_{\substack{(i_1,\cdots,i_q) \\ \text{pairwise distinct}}} \lambda_{i_1}^2 \cdots \lambda_{i_{q-1}}^2 \underbrace{\sum_{i_q \notin \{i_1,\cdots,i_{q-1}\}} \lambda_{i_q}^2}_{\geq r_q(A)} \geq \frac{(q-1)!}{q!} R_{q-1}(A) r_q(A). \end{aligned}$$

□

We also have the following lemma establishing that the two quantities $r_q(A)$ and $\mathcal{D}_q(A)$ coincide.

Lemma 4.

$$\mathcal{D}_q(A) = r_q(A). \tag{17}$$

Proof. This result is actually a consequence of the Eckart–Young–Mirsky Theorem (for the Frobenius norm). One may consult [5, 7.4.15] for more details. To state more precisely this result we need a matrix $M \in \mathcal{M}_{m,n}(\mathbb{R})$ of rank k with singular value decomposition $M = V\Sigma^\intercal W$ where V, W are orthogonal matrices of sizes respectively $m \times m$ and $n \times n$ and Σ is a rectangular diagonal matrix of size $m \times n$ with non negative real numbers on the diagonal $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_k > 0$ and $\sigma_{k+1} = \sigma_{k+2} = \cdots = \sigma_{\min(n,m)} = 0$. For $k_1 < k$ we denote by Σ_1 the same as Σ except that only $\sigma_1, \cdots, \sigma_{k_1}$ are used and all the remaining diagonal entries being zero. Then,

$$\min_{B \in \mathcal{M}_{m,n}(\mathbb{R}), \mathrm{rank}(B) \leq k_1} \|M - B\| = \|M - V\Sigma_1{}^\intercal W\| = \|\Sigma - \Sigma_1\| = \sqrt{\sum_{i=k_1+1}^{k} \sigma_i^2}.$$

Placing ourselves in the particular case of a symmetric square matrix M whose spectrum is ordered by decreasing absolute values, one deduces that $M^2 = V\Sigma^\intercal W W^\intercal \Sigma^\intercal V = V\, \Sigma^2\, {}^\intercal V$. From that we deduce $\sigma_1 = |\lambda_1|, \sigma_2 = |\lambda_2|, \cdots, \sigma_k = |\lambda_k|$ and that $\sigma_i = \lambda_i = 0$ for $i > k$. The result follows. □

2 Proof of the Main Theorem

Let us introduce the following family of mathematical properties.

$\mathcal{P}(m) :=$ For every $q \geq 1$ there exists $\delta_q > 0$ and $C_q > 0$ such that for every F in $\mathcal{W}_m$:

$$d_{FM}(F, \mathcal{N}(0,1)) \leq \delta_q \Rightarrow \mathbb{E}\left[\frac{1}{\Gamma[F,F]^q}\right] \leq C_q.$$

In this section we shall prove that $\mathcal{P}(2)$ is true and that for every $m \geq 3$ we have $\mathcal{P}(m-1) \Rightarrow \mathcal{P}(m)$.

2.1 A key recurrence lemma

Let us start with the following Lemma relying on a Fourier-Laplace identity.

Lemma 5. *Let us assume that m is an integer larger than three and that the Theorem 1 is true for Wiener chaoses of order less or equal than $m-1$. Let $p \geq 1$, there exists $\delta_p > 0$ and $C_p > 0$ such that $\forall k \geq 1$, $\forall C \in \mathcal{M}_{k,1}(\mathcal{W}_{m-2})$, $\forall Y \sim \mathcal{N}(0, Id_k) \perp\!\!\!\perp C$,*

$$d_{FM}(^{\intercal}CY, \mathcal{N}(0,1)) \leq \delta_p \Rightarrow \forall \epsilon > 0,\, \mathbb{P}(\|C\|_2 \leq \epsilon) \leq C_p \epsilon^p \tag{18}$$

Proof. We compute the Laplace transform of $\|C\|_2$ which via a Laplace-Fourier identity (see subsection 1.1.3) leads for all real number s to

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\frac{s^2}{2}\sum_{i=1}^{k} C_i^2\right)\right] &= \mathbb{E}\left[\exp\left(is\sum_{i=1}^{k} C_i Y_i\right)\right] \\ &= \mathbb{E}\left[\exp\left(is^{\intercal}CY\right)\right].\end{aligned}$$

Since $Y \perp\!\!\!\perp C$, $^{\intercal}CY$ may be seen as an element of the $m-1$-th Wiener chaos. Besides, since we assumed that the Theorem 1 is true at the order $m-1$ then the content of remark 3 applies. We can conclude as in the end of the step 1 of the proof of proposition 2. □

We shall generalize in the sequel the previous lemma, to do so we shall need the next simple lemma of discretization of the $q-1$-dimensional Euclidean sphere $\mathcal{S}^{q-1}$.

Lemma 6. *For all $q \geq 2$ and $N \geq 1$ there exists $C_q > 0$ (not depending on N) and $\mathcal{S}^{q-1,N} \subset \mathcal{S}^{q-1}$ such that $Card\left(\mathcal{S}^{q-1,N}\right) \leq C_q N^q$ and*

$$\forall a \in \mathcal{S}^{q-1},\, \exists \tilde{a} \in \mathcal{S}^{q-1,N},\, \text{such that}\, \|a - \tilde{a}\|_2 \leq \frac{C_q}{N}.$$

Proof. We fix an integer $N > 0$, for any $I = (i_1, i_2, \cdots, i_q) \in [\![-N, N]\!]^q$ we set $b_I = (\frac{i_1}{N}, \frac{i_2}{N}, \cdots, \frac{i_q}{N})$. It is clear that for any $a \in \mathcal{S}^{q-1}$ we may find $I \in [\![-N, N]\!]^q$ such that $\|b_I - a\|_\infty \leq \frac{1}{N}$. Hence $|\|b_I\|_2 - 1| \leq \|b_I - a\|_2 \leq \sqrt{q}\|b_I - a\|_\infty \leq \frac{\sqrt{q}}{N}$. For $N \geq 2\sqrt{q}$ we must have $\|b_I\|_2 \geq \frac{1}{2}$ and setting $a_I = \frac{b_I}{\|b_I\|_2}$ we must have

$$\|a_I - a\|_2 \leq 2\,\|b_I - a\|b_I\|_2\|_2 \leq 2\|b_I - a\|_2 + 2|1 - \|b_I\|_2| \leq \frac{4\sqrt{q}}{N}.$$

Denoting by $S^{q-1,N}$ the set of $\frac{b_I}{\|b_I\|_2} = a_I$ for $I \in [\![-N, N]\!]^q/\{0, \cdots, 0\}$ we have proved that, provided that $N \geq 2\sqrt{q}$, for any $a \in \mathcal{S}^{q-1}$ we may find $a_I \in \mathcal{S}^{q-1,N}$ such that $\|a - a_I\| \leq \frac{4\sqrt{q}}{N}$. Besides Card $\left(\mathcal{S}^{q-1,N}\right) \leq (2N+1)^q$. □

The next lemma is a key element of our proof. It enables us to proceed by induction on the order of the chaos m.

Lemma 7. *In this lemma, we fix $m \geq 3$ and $q \geq 1$. Besides, we assume that we have established the main theorem 1 in the case of Wiener chaoses of order less or equal than $m-1$.*

For all $p \geq 1$, there is a positive constant $\delta_{p,q}$ such that for any $k \geq 1$, for any random matrix $M \in \mathcal{M}_{k,q}(\mathcal{W}_{m-2})$ and any independent random vector $Y \sim \mathcal{N}(0, Id_k)$:

$$d_{FM}(^{\intercal}MY, \mathcal{N}(0, Id_q)) \leq \delta_{p,q} \Rightarrow \forall \epsilon > 0,\, \mathbb{P}(\mathcal{D}_q(M) \leq \epsilon) \leq C_{p,q}\epsilon^p.$$

Proof. We divide our proof in several steps.

Step 1: Proving that $\{\mathcal{D}_q(M) \le \epsilon\} \subset \{\inf_{a\in\mathcal{S}^{q-1}} \|M^\intercal a\| \le \sqrt{\epsilon}\}$.

Let us place ourselves on the event $\{\mathcal{D}_q(M) \le \epsilon\}$ and let us consider a (random) matrix B of rank less or equal than $q-1$ such that $\mathcal{D}_q(M) = \|M - B\|^2$. Let $a = (a_1, a_2, \cdots, a_q)$ be an element of the $q-1$-dimensional Euclidean sphere. Denoting by M_i and B_i the columns of M and B for $i \in \{1, \cdots, q\}$ we may write

$$
\begin{aligned}
\|M^\intercal a - B^\intercal a\|_2 &= \|\sum_{i=1}^{q} a_i(M_i - B_i)\|_2 \\
&\le \sum_{i=1}^{q} |a_i| \|M_i - B_i\|_2 \\
&\overset{\text{CS}}{\le} \sqrt{\sum_{i=1}^{q} \|M_i - B_i\|_2^2} = \sqrt{\mathcal{D}_q(M)}.
\end{aligned}
$$

This in particular implies that

$$
\inf_{a\in\mathcal{S}^{q-1}} \|M^\intercal a\| \le \underbrace{\inf_{a\in\mathcal{S}^{q-1}} \|B^\intercal a\|}_{=0,\ \text{since rank}(B)\le q-1} + \sqrt{\mathcal{D}_q(M)}.
$$

From that we deduce that

$$
\{\mathcal{D}_q(M) \le \epsilon\} \subset \{\inf_{a\in\mathcal{S}^{q-1}} \|M^\intercal a\| \le \sqrt{\epsilon}\}. \tag{19}
$$

Step 2: Discretizing the Euclidean sphere $\mathcal{S}^{q-1}$

Let N be an integer to be chosen later. We apply lemma 6, for any $a \in \mathcal{S}^{q-1}$ and $\tilde{a} \in \mathcal{S}^{q-1,N}$ such that $\|a - \tilde{a}\|_2 \le \frac{C_q}{N}$, we may write:

$$
\begin{aligned}
\|M^\intercal a - M^\intercal \tilde{a}\|_2 &\le \sum_{i=1}^{q} |a_i - \tilde{a}_i| \|M_i\|_2 \le \|a - \tilde{a}\|_2 \|M\| \le \frac{C_q}{N}\|M\|, \\
\inf_{a\in\mathcal{S}^{q-1,N}} \|M^\intercal a\|_2 &\le \inf_{a\in\mathcal{S}^{q-1}} \|M^\intercal a\|_2 + \frac{C_q}{N}\|M\|.
\end{aligned}
$$

From (19) we deduce

$$
\{D_q(M) \le \epsilon\} \subset \{\inf_{a\in\mathcal{S}^{q-1,N}} \|M^\intercal a\|_2 \le \sqrt{\epsilon} + \frac{C_q K}{N}\} \cup \{\|M\| > K\}. \tag{20}
$$

By the union bound and Markov inequality for any exponent $r > 0$ we obtain

$$
\mathbb{P}\left(\mathcal{D}_q(M) \le \epsilon\right) \le \sum_{a\in\mathcal{S}^{q-1,N}} \mathbb{P}\left(\|M^\intercal a\|_2 \le \sqrt{\epsilon} + \frac{C_q K}{N}\right) + \underbrace{\frac{C_r}{K^r}\mathbb{E}\left[\|M\|^2\right]^{\frac{r}{2}}}_{\text{Markov+hypercontractivity}}.
$$

We make the following choices

- $N = \frac{1}{\epsilon}$,
- $K = \frac{1}{\sqrt{\epsilon}}$,
- $r = 2p$,

so that for some constants $C_p, C_q > 0$:

$$\mathbb{P}\left(\mathcal{D}_q(M) \leq \epsilon\right) \leq \sum_{a_I \in \mathcal{S}^{q-1,N}} \mathbb{P}\left(\|M^\intercal a\|_2 \leq C_q\sqrt{\epsilon}\right) + C_p \epsilon^p \mathbb{E}\left[\|M\|^2\right]^p.$$

Step 3: Using lemma 5 to handle $\mathbb{P}\left(\|M^\intercal a\|_2 \leq C_q\sqrt{\epsilon}\right)$.

It turns out that $a^\intercal MY$ may be seen as a Wiener chaos of order $m-1$ and $d_{\mathrm{FM}}\left(a^\intercal MY, \mathcal{N}(0,1)\right) \leq d_{\mathrm{FM}}\left({}^\intercal MY, \mathcal{N}(0, I_q)\right)$. Hence one may use the lemma 5 with $C = M^\intercal a$. Provided that $d_{\mathrm{FM}}\left({}^\intercal MY, \mathcal{N}(0, I_q)\right)$ is less than $\delta_{p,q}$ small enough, there exists $C_{p,q} > 0$ (which may change from line to line) such that

$$\forall a \in \mathcal{S}^{q-1},\ \forall \epsilon > 0,\ \mathbb{P}\left(\|M^\intercal a\|_2 \leq C_q\sqrt{\epsilon}\right) \leq C_{p,q}\epsilon^{p+q}.$$

Step 4: Final argument.

Since $\mathrm{Card}\left(\mathcal{S}^{q-1,N}\right) \leq \frac{C_q}{\epsilon^q}$ we may write

$$\begin{aligned}\mathbb{P}\left(\mathcal{D}_q(M) \leq \epsilon\right) &\leq \sum_{a \in \mathcal{S}^{q-1,N}} \mathbb{P}\left(\|M^\intercal a\|_2 \leq C_q\sqrt{\epsilon}\right) + C_p \epsilon^p \mathbb{E}\left[\|M\|^2\right]^p \\ &\leq C_{p,q}\frac{1}{\epsilon^q}\epsilon^{p+q} + C_p \epsilon^p \mathbb{E}\left[\|M\|^2\right]^p.\end{aligned}$$

Finally, there exists a constant $C_m > 0$ such that, whenever $d_{\mathrm{FM}}\left({}^\intercal MY, \mathcal{N}(0, \mathrm{Id}_q)\right) \leq 1$, $\mathbb{E}\left[\|M\|^2\right] \leq C_m$. Indeed, a consequence of the hypercontractivity on Wiener chaoses is that the convergence in law implies the convergence of every moments. As a conclusion, there exists $\delta_p > 0$ small enough and some constant $C_{p,q}$ such that

$$d_{\mathrm{FM}}\left({}^\intercal MY, \mathcal{N}(0, I_q)\right) < \delta_p \Rightarrow \forall \epsilon > 0, \mathbb{P}\left(\mathcal{D}_q(M) \leq \epsilon\right) \leq C_{p,q}\epsilon^p, \tag{21}$$

which achieves the proof.

□

2.2 The induction

In this section, we implement our induction and terminate the proof of the main Theorem 1. We divide our argumentation in several distinct steps. Relying on Proposition 2, we give us $q \geq 1$ and we want to prove that $\widetilde{F_n} \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0,1)$.

Step 1 : initialization for $m = 2$

Let F be any element of $\mathcal{W}_2^{(0)}$. It is well known that for some integer K

$$F \overset{\text{law}}{=} \sum_{k=1}^{K} \lambda_k (N_k^2 - 1),$$

where $(\lambda_k)_{1\le k\le K}$ are the eigenvalues of the matrix of the associated quadratic form denoted by A. Hence, one obtains

$$\begin{aligned} \left|\mathbb{E}\left[e^{itF}\right]\right| &= \prod_{i=1}^{K} \frac{1}{\left|\sqrt{1-2it\lambda_i}\right|} \\ &= \prod_{i=1}^{K} \frac{1}{(1+4t^2\lambda_i^2)^{\frac14}} \\ &\le \frac{1}{t^{\frac{q}{2}}} \frac{1}{\mathcal{R}_q(A)^{\frac14}}. \end{aligned} \tag{22}$$

As a result, when $m = 2$, in order to establish that $\widetilde{F_n} \xrightarrow[n\to\infty]{\mathcal{F}^q} \mathcal{N}(0,1)$ for every q, it is sufficient to bound from below the quantities $\mathcal{R}_q(\widetilde{A_n})$ and $\widetilde{A_n}$ is deterministic in this case with spectrum $(\lambda_{i,n})_{1\le i\le 2k_n}$. Relying on the equation (16), one is left to bound from below $r_q(\widetilde{A_n})$ for every q. However, $\mathbb{E}\left[\widetilde{F_n}^2\right] \to 1$ and $\mathbb{E}\left[\widetilde{F_n}^4\right] \to 3$ which in turns is equivalent to the fact that $\sum_{i=1}^{k_n} \lambda_{i,n}^2 \to \frac12$ and $\sum_{i=1}^{k_n} \lambda_{i,n}^4 \to 0$. Hence, $\rho(\widetilde{A_n}) \to 0$ and $r_q(\widetilde{A_n}) \ge \frac12 - q\rho(A_n)^2$ which is bounded from below for n sufficiently large.

Step 2 : relating with the spectrum of a random quadratic form

We proceed by induction on m, we assume that $m \ge 3$ and that the theorem 1 is true for Wiener chaoses of order $m-1$. We refer to Definition 3 for the special form of $\widetilde{F_n}$, A_n and $\widetilde{A_n}$. In particular, $\widetilde{F_n}$ may be seen as a second Wiener chaos induced by an independent and random matrix $\widetilde{A_n}$. We point out that in the previous step $\widetilde{A_n}$ was deterministic only because $m = 2$. Using the equation (22) and taking the expectation with respect to $\widetilde{A_n}$, it is sufficient to establish

$$\forall q \ge 1,\ \limsup_n \mathbb{E}\left[\frac{1}{\mathcal{R}_q(\widetilde{A_n})^{\frac14}}\right] < \infty.$$

However, using (16), (17), and the remark 4, one has

$$\mathcal{R}_{2q}(\widetilde{A_n}) \ge \frac{1}{(2q)!} r_{2q}(\widetilde{A_n})^q = \frac{2^q}{(2q)!} r_q(A_n)^q = \frac{2^q}{(2q)!} \mathcal{D}_q(A_n)^q.$$

In particular, it is sufficient to prove that

$$\forall q \ge 1, \forall p \ge 1,\ \limsup_n \mathbb{E}\left[\frac{1}{\mathcal{D}_q(A_n)^p}\right] < \infty.$$

This will be a consequence of getting estimates of $\mathbb{P}\left(\mathcal{D}_q(A_n) \le \epsilon\right)$ with polynomial decay in ϵ at arbitrarily high order.

Step 3 : A kind of compress sensing argument

Let us fix $q \geq 1$ and a deterministic matrix $\mathcal{X} \in \mathcal{M}_{k_n,q}(\mathbb{R})$ that we will carefully choose later. It is then sufficient to establish that for every $p \geq 1$, there exists $N_{p,q} \in \mathbb{N}$ and $C_{p,q} > 0$ such that for every $n \geq N_{p,q}$ and for every $\epsilon > 0$ we have $\mathbb{P}\left(\mathcal{D}_q(A_n) \leq \epsilon\right) \leq C_{p,q}\epsilon^p$. We then write for $K > 0$ to be chosen later and for any exponent $p \geq 1$:

$$\begin{aligned}
\mathbb{P}\left(\mathcal{D}_q(A_n) \leq \epsilon\right) &\leq \mathbb{P}\left(\mathcal{D}_q(A_n) \leq \epsilon, \mathcal{D}_q(A_n\mathcal{X}) \leq K\mathcal{D}_q(A)\right) \\
&+ \mathbb{P}\left(\mathcal{D}_q(A_n\mathcal{X}) \geq K\mathcal{D}_q(A_n)\right) \\
&\leq \mathbb{P}\left(\mathcal{D}_q(A_n\mathcal{X}) \leq K\epsilon\right) + \frac{1}{K^p}\mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_q(A_n)^p}\right] \\
&\overset{K=\epsilon^{-\frac{1}{2}}}{\leq} \mathbb{P}\left(\mathcal{D}_q(A_n\mathcal{X}) \leq \epsilon^{\frac{1}{2}}\right) + \epsilon^{\frac{p}{2}}\mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_q(A_n)^p}\right].
\end{aligned}$$

Step 4 : choosing a good $\mathcal{X}$ via the probabilistic method

(i) First of all, let us handle the term $\mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_q(A_n)^p}\right]$. Let us denote by $\gamma_{k_n,q}$ the product of standard Gaussian measures on $\mathbb{R}^{k_n} \times \mathbb{R}^q$ which naturally endows the set of matrices of $\mathcal{M}_{k_n,q}(\mathbb{R})$ with a standard Gaussian measure. If A is any matrix in $\mathcal{M}_{k_n}(\mathbb{R})$, and if B is a matrix of rank less than $q-1$ such that $\|A - B\| = \mathcal{D}_q(A)$, then we have

$$\begin{aligned}
\int_{\mathcal{M}_{k_n,q}(\mathbb{R})} \mathcal{D}_q(A\mathcal{X})^p d\gamma_{k_n,q}(\mathcal{X}) &\leq \mathbb{E}_X\left[\|AX - BX\|^{2p}\right] \\
&\leq \underbrace{C_p}_{\text{hypercontractivity}} \mathbb{E}_X\left[\|AX - BX\|^2\right]^p,
\end{aligned}$$

using that for any matrix $X \in \mathcal{M}_{k_n,q}(\mathbb{R})$ we have $\operatorname{rank}(BX) \leq q-1$. Above, X is a random matrix of size $k_n \times q$ whose coefficients are independent standard Gaussian. Besides, for any matrix $M \in \mathcal{S}_{k_n}(\mathbb{R})$ we have

$$\begin{aligned}
\mathbb{E}_X\left[\|MX\|^2\right] &= \mathbb{E}\left[\operatorname{Tr}\left(\underbrace{{}^{\intercal}X}_{q\times k_n}\underbrace{{}^{\intercal}MM}_{k_n\times k_n}\underbrace{X}_{k_n\times q}\right)\right] \\
&= \mathbb{E}\left[\operatorname{Tr}\left(\underbrace{{}^{\intercal}MM}_{k_n\times k_n}\underbrace{X}_{k_n\times q}\underbrace{{}^{\intercal}X}_{q\times k_n}\right)\right] \\
&= \operatorname{Tr}\left({}^{\intercal}MM\underbrace{\mathbb{E}\left[X{}^{\intercal}X\right]}_{=q\mathrm{Id}_{k_n}}\right) = q\|M\|^2.
\end{aligned}$$

The last equality proceeds from $\mathbb{E}\left[X{}^{\intercal}X(i,j)\right] = \sum_{k=1}^{q}\mathbb{E}\left[X(i,k)X(j,k)\right] = q\ \delta_{i,j}$. This in particular guarantees that $\mathbb{E}_X\left[\|AX - BX\|^2\right] = q\|A - B\|^2 = q\mathcal{D}_q(A)$. So we obtained that for some constant $C_{p,q} > 0$,

$$\int_{\mathcal{M}_{k_n,q}(\mathbb{R})} \mathcal{D}_q(A\mathcal{X})^p d\gamma_{k_n,q}(\mathcal{X}) \leq C_{p,q}\mathcal{D}_q(A)^p.$$

Applying this estimate to $A = A_n$ and taking the expectation we deduce

$$\begin{aligned} &\int_{\mathcal{M}_{k_n,q}(\mathbb{R})} \mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_p(A_n)^p}\right] d\gamma_{k_n,q}(\mathcal{X}) \\ = \ & \mathbb{E}\left[\frac{1}{\mathcal{D}_q(A_n)^p}\int_{\mathcal{M}_{k_n,q}(\mathbb{R})} \mathcal{D}_q(A_n\mathcal{X})^p d\gamma_{k_n,q}(\mathcal{X})\right] \\ \leq \ & \mathbb{E}\left[\frac{1}{\mathcal{D}_q(A_n)^p} \times C_{p,q}\mathcal{D}_q(A_n)^p\right] = C_{p,q}. \end{aligned}$$

As a result, we obtain that by Markov inequality that

$$\begin{aligned} &\gamma_{k_n,q}\left(\left\{\mathcal{X} \in \mathcal{M}_{k_n,q}(\mathbb{R}) \middle| \mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_p(A_n)^p}\right] \geq 3C_{p,q}\right\}\right) \\ \leq \ & \frac{1}{3C_{p,q}}\int_{\mathcal{M}_{k_n,q}(\mathbb{R})} \mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_p(A_n)^p}\right] d\gamma_{k_n,q}(\mathcal{X}) \\ \leq \ & \frac{1}{3}. \end{aligned} \tag{23}$$

(ii) Let us now handle the term $\mathbb{P}\left(\mathcal{D}_q(A_n\mathcal{X}) \leq \sqrt{\epsilon}\right)$. For any deterministic $\mathcal{X}$ of size $k_n \times q$, we get that $A_n\mathcal{X}$ is a matrix of size $k_n \times q$ whose entries belong to the Wiener chaos of order $m-2$. Besides, as we proceed by induction we have assumed that the main theorem 1 holds true for Wiener chaoses of order $m-1$. It is then licit to use lemma 7, asserting that there exists positive constants $\delta_{p,q} > 0, C_{p,q} > 0$ such that

$$d_{\mathrm{FM}}\left({}^{\intercal}(A_n\mathcal{X})Y, \mathcal{N}(0, \mathrm{Id}_q)\right) \leq \delta_{p,q} \Rightarrow \forall \epsilon > 0,\ \mathbb{P}\left(\mathcal{D}_q\left({}^{\intercal}(A_n\mathcal{X})\right) \leq \epsilon\right) \leq C_{p,q}\epsilon^p.$$

For that, we now use lemma 2 applied to ${}^{\intercal}(A_n\mathcal{X})Y$. By the definition of the convergence in probability we may derive that:

$$\begin{aligned} &\gamma_{k_n,q}\left(\left\{\mathcal{X} \in \mathcal{M}_{k_n,q}(\mathbb{R}) \,|\, d_{\mathrm{FM}}\left({}^{\intercal}(A_n\mathcal{X})Y, \mathcal{N}(0, \mathrm{Id}_q)\right) > \delta_{p,q}\right\}\right) \\ &\xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Hence, there exists $N_{p,q} \geq 1$ such that for every $n \geq N_{p,q}$ we have

$$\begin{aligned} &\gamma_{k_n,q}\left(\left\{\mathcal{X} \in \mathcal{M}_{k_n,q}(\mathbb{R}) \,|\, d_{\mathrm{FM}}\left({}^{\intercal}(A_n\mathcal{X})Y, \mathcal{N}(0, \mathrm{Id}_q)\right) > \delta_{p,q}\right\}\right) \\ &\leq \frac{1}{3}. \end{aligned} \tag{24}$$

(iii) Let $n \geq N_{p,q}$, the equations (23) and (24) provide two events of probability measure less than $\frac{1}{3}$. The probability that none of these events is realized is then greater than $\frac{1}{3}$ and hence positive. As a result, one may find $\mathcal{X} \in \mathcal{M}_{k_n,q}(\mathbb{R})$ such that we have simultaneously

* $\mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_p(A_n)^p}\right] \leq 3C_{p,q}$,
* $d_{\mathrm{FM}}\left({}^\intercal(A_n\mathcal{X})Y, \mathcal{N}(0,\mathrm{Id}_q)\right) \leq \delta_{p,q}$ and thus

$$\forall\epsilon > 0,\ \mathbb{P}\left(\mathcal{D}_q\left({}^\intercal(A_n\mathcal{X})\right) \leq \epsilon\right) \leq \epsilon^p.$$

Coming back to the final inequality of the step 2, for this particular choice of $\mathcal{X}$, we finally get for $n \geq N_{p,q}$ that

$$\begin{aligned} \mathbb{P}\left(r_q(A_n) \leq \epsilon\right) &\leq \mathbb{P}\left(\mathcal{D}_q(A_n\mathcal{X}) \leq \sqrt{\epsilon}\right) + \epsilon^{\frac{p}{2}}\mathbb{E}\left[\frac{\mathcal{D}_q(A_n\mathcal{X})^p}{\mathcal{D}_q(A_n)^p}\right] \\ &\leq C_{p,q}\epsilon^{\frac{p}{2}}, \end{aligned}$$

which by the content of step 2 achieves the proof.

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